

Quasi-conformal actions, quaternionic discrete series and twistors: $SU(2, 1)$ and $G_{2(2)}$

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ABSTRACT: Quasi-conformal actions were introduced in the physics literature as a generalization of the familiar fractional linear action on the upper half plane, to Hermitian symmetric tube domains based on arbitrary Jordan algebras, and further to arbitrary Freudenthal triple systems. In the mathematics literature, quaternionic discrete series unitary representations of real reductive groups in their quaternionic real form were constructed as degree 1 cohomology on the twistor spaces of symmetric quaternionic-Kähler spaces. These two constructions are essentially identical, as we show explicitly for the two rank 2 cases $SU(2, 1)$ and $G_{2(2)}$. We obtain explicit results for certain principal series, quaternionic discrete series and minimal representations of these groups, including formulas for the lowest K -types in various polarizations. We expect our results to have applications to topological strings, black hole micro-state counting and to the theory of automorphic forms.

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1. Introduction and Summary

Despite much recent progress, classifying the unitary representations of a real reductive group G remains a challenging task, which has only been addressed in a few, typically low-rank cases, including for example $SL(2, \mathbb{R})$ [1, 2, 3], $SL(3, \mathbb{R})$ [4, 5], $SU(2, 1)$ [6], $SU(2, 2)$ [7] and $G_{2(2)}$ [8]. Even in well-charted cases, the available mathematical description is often not directly useful to physicists, who are in general more interested in explicit differential operator realizations than abstract classifications. The goal of this paper is to give an explicit description of aspects of principal and discrete series representations (and continuations of the discrete series) which arise when G is a quaternionic real form of a semi-simple group. For the sake of explicitness, we restrict to the rank 2 case – so $G = SU(2, 1)$ or $G = G_{2(2)}$. Quaternionic real forms arise as symmetries of supergravity theories with 8 supercharges in three dimensions [9, 10, 11, 12, 13], and therefore as spectrum-generating symmetries for black holes in four dimensional supergravity; a detailed discussion of this relation, which indeed motivated our interest in the first place, can be found in [14, 15, 16] and references therein.

The simplest discrete series representations¹ arise when $K = U(1) \times M$, so that G/K is a Hermitian symmetric domain. The simplest example is $G/K = SL(2, \mathbb{R})/U(1)$: G acts holomorphically on the upper half-plane by fractional linear transformations $\tau \rightarrow (a\tau + b)/(c\tau + d)$, which preserve the Kähler potential $K = -\log[(\tau - \bar{\tau})]$ up

¹Discrete series representations arise as discrete summands in the spectral decomposition of $L^2(G)$ under the left action of G . A basic result of Harish-Chandra [17, 18] states that G admits discrete series representations if and only if its maximal compact subgroup K has the same rank as G itself.

to Kähler transformations. This construction can be extended to all Hermitian symmetric tube domains using the language of Jordan algebras. Indeed, consider the “upper half plane” $\tau \in J + iJ^+$, where J is a Euclidean Jordan algebra J of degree n and J^+ is the domain of positivity of J , equipped with the Kähler potential $K = -\log \mathcal{N}(\tau - \bar{\tau})$, where \mathcal{N} is the norm form of J ². The corresponding metric is invariant under a non-compact group $G = \text{Conf}(J)$, the “conformal group” associated to J , acting holomorphically by generalized fractional linear transformations on τ [24, 25]. The resulting space is the Hermitian symmetric tube domain $G/K = \text{Conf}(J)/\widetilde{\text{Str}}(J)$, where $\widetilde{\text{Str}}(J) = U(1) \times \widetilde{\text{Str}}_0(J)$ is the compact real form of the structure group³ of J , which coincides with the maximal compact subgroup of G . The action of G on sections of holomorphic vector bundles over G/K leads to the holomorphic discrete series representations.

The next simplest case, of interest in this paper, arises when G is in its quaternionic real form, such that $K = SU(2) \times M$, and its Lie algebra decomposes as $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{M} \oplus (2, V)$, where V is a pseudo-real representation of M . The symmetric space G/K , of real dimension $4d$, is now a quaternionic-Kähler space, and does not generally admit a G -invariant complex structure. The twistor space $\mathcal{Z} = G/U(1) \times M$, a bundle over G/K with fiber $\mathbb{CP}^1 = SU(2)/U(1)$, does however carry a G -invariant complex structure. In [26] this complex structure was exploited to construct a family of representations π_k of G , labeled by $k \in \mathbb{Z}$: namely, π_k is the sheaf cohomology $H^1(\mathcal{Z}, \mathcal{O}(-k))$ of a certain line bundle $\mathcal{O}(-k)$ over \mathcal{Z} .⁴ π_k is a representation of G , with Gelfand-Kirillov (functional) dimension $2d+1$, equal to the complex dimension of \mathcal{Z} . For $k \geq 2d+1$, π_k are discrete series unitary representations of G , called “quaternionic discrete series.”

The π_k for $k < 2d+1$ are also of interest. In [26] special attention was paid to quaternionic groups of type $G_2, D_4, F_4, E_6, E_7, E_8$, such that $d = 3f+4$ with $f = -\frac{2}{3}, 0, 1, 2, 4, 8$. For these groups, it was shown that π_k is irreducible and unitarizable even for $k \geq d+1$ (although it no longer belongs to the quaternionic discrete series for $k < (2d+1)$). Moreover, for selected smaller values of k , namely $k = 3f+2, k = 2f+2$

²The generalized upper half planes associated to Jordan algebras are sometimes called Köcher half-spaces since Köcher pioneered their study [19]. In particular, he introduced the linear fractional groups of Jordan algebras [20], which were interpreted as conformal groups of generalized spacetimes defined by Jordan algebras and extended to Jordan superalgebras in the physics literature [21, 22, 23]. The terminology stems from the well-known action of the conformal group $SO(4, 2)$ on Minkowski spacetime, which arises when J is the Jordan algebra of 2×2 Hermitian matrices [21]. Choosing $J = \mathbb{R}$ leads instead to our original example $G/K = SL(2, \mathbb{R})/U(1)$.

³The structure group $\text{Str}(J)$ leaves the norm form \mathcal{N} of the Jordan algebra J invariant up to an overall scaling. It decomposes as an Abelian factor \mathbb{R} times the reduced structure group $\text{Str}_0(J)$ which leaves the norm invariant.

⁴For a definition of sheaf cohomology see *e.g.* [27].

and $k = f + 2$, π_k was shown to be reducible but to admit a unitarizable submodule π'_k , of smaller Gelfand-Kirillov dimension, $2d = 6f + 8$, $5f + 6$, $3f + 5 = d + 1$, respectively. The smallest of these representations is the “minimal” or “ladder” representation of G ; the latter name refers to the structure of its K -type decomposition⁵; a review of minimal representations can be found in [28].

Independently of these mathematical developments, it was shown in [25] that the conformal realization of the group $\text{Conf}(J)$ attached to a Jordan algebra J of degree 3, preserving a generalized light-cone $\mathcal{N}_3(\tau - \bar{\tau}) = 0$, could be extended to an action of a larger non-compact group, preserving a “quartic light-cone”

$$N_4 \equiv I_4(\Xi - \bar{\Xi}) + (\alpha - \bar{\alpha} + \langle \Xi, \bar{\Xi} \rangle)^2 = 0, \quad (1.1)$$

where $\Xi = (\xi^I, \tilde{\xi}_I) = (\xi^0, \xi^\Lambda, \tilde{\xi}_\Lambda, \tilde{\xi}_0)$ is an element of the Freudenthal triple system $\mathcal{F} = \mathbb{C} \oplus J_{\mathbb{C}} \oplus J_{\mathbb{C}} \oplus \mathbb{C}$ associated to J , $\langle \Xi, \bar{\Xi} \rangle$ a symplectic pairing invariant under the linear action of $\text{Conf}(J)$ on Ξ , I_4 a quartic polynomial invariant under this same action⁶ and α an additional complex variable of homogeneity degree two. This larger group was called the “quasi-conformal group” $\text{QConf}(J)$ attached to the Jordan algebra J , or more appropriately to the Freudenthal triple system \mathcal{F} ; its geometric action on (Ξ, α) was called the “quasi-conformal realization”. When J is Euclidean, $\text{QConf}(J)$ is a non-compact group in its quaternionic real form; other real forms can be similarly obtained from Jordan algebras of indefinite signature [25]. Moreover, it was observed in [29] for $G = E_{8(8)}$, and generalized to other simple groups in [30, 31, 32], that this quasi-conformal action on $2d + 1$ variables could be reduced to a representation on functions of $d + 1$ variables, obtained by first adding one more variable (symplectizing) and then quantizing the resulting $2d + 2$ -dimensional symplectic space. This smaller representation was identified as the minimal representation of $G = \text{QConf}(J)$.

⁵We remind the reader of a few definitions. Suppose ρ is a representation of a real Lie group G on a vector space V . Let V_K denote the space of K -finite vectors in V (i.e. those which generate a finite-dimensional subspace of V under the action of the maximal compact subgroup $K \subset G$). Then any representation of K which occurs in V_K is called a K -type of ρ . Parameterizing the K -types by their highest weights μ , a “lowest K -type” is one with the minimal value of $\|\mu + 2\rho_K\|^2$, where ρ_K is half the sum of positive roots of K . A discrete series representation always has a unique lowest K -type. If the lowest K -type is the trivial representation, then it is also called the spherical vector and the representation ρ is deemed spherical. The Gelfand-Kirillov dimension measures the growth of the multiplicities of the K -types; morally it counts the number of variables x_i needed to realize V as a space of functions. Finally, if all K -types in ρ occur with multiplicity 1 and lie along a ray in the weight space of K , then ρ is called a ladder representation.

⁶ I_4 is expressible in terms of \mathcal{N}_3 via $8I_4(\Xi) = \left(\xi^0 \tilde{\xi}_0 - \xi^\Lambda \tilde{\xi}_\Lambda\right)^2 - 4\xi_\Lambda^\# \tilde{\xi}_\Lambda^\# + 4\xi^0 \mathcal{N}_3(\xi^\Lambda) + 4\tilde{\xi}_0 \mathcal{N}_3(\tilde{\xi}_\Lambda)$, where $\xi_\#^\#$ is related to ξ by the (quadratic) adjoint map, defined by $(\xi^\#)^\# = \mathcal{N}(\xi)\xi$.

Let us now comment briefly on the physics. Euclidean Jordan algebras J of degree three made their appearance in supergravity a long time ago [9, 10, 33]. Maxwell-Einstein supergravity theories with $\mathcal{N} = 2$ supersymmetry in $D = 5$ and a symmetric moduli space G/H such that G is a symmetry group of the action are in one-to-one correspondence with Euclidean Jordan algebras J of degree three. Their symmetry group in $D = 5$ is simply the reduced structure group $\text{Str}_0(J)$ of J . Upon reduction to $D = 4$ and $D = 3$, the symmetry groups are extended to the conformal $\text{Conf}(J)$ and quasi-conformal groups $\text{QConf}(J)$, respectively. The corresponding moduli spaces are given by the quotient of the respective symmetry group by its maximal compact subgroup, and are special real, special Kähler and quaternionic-Kähler manifolds, respectively [9, 10, 13]. An explicit description of the quasi-conformal action $\text{QConf}(J)$ of $D = 3$ Maxwell-Einstein supergravity theories with symmetric target spaces was obtained in [31]. Finally, it was observed in [15] that the minimal unitary representation of $\text{QConf}(J)$ is closely related to the vector space to which the topological string partition function naturally belongs.

The minimal representation has also appeared in another physical context: in [34], a connection between the harmonic superspace (HSS) formulation of $\mathcal{N} = 2$, $d = 4$ supersymmetric quaternionic Kähler sigma models that couple to $\mathcal{N} = 2$ supergravity and the minimal unitary representations of their isometry groups was established. In particular, for $\mathcal{N} = 2$ sigma models with quaternionic symmetric target spaces of the form $\text{QConf}(J)/\widetilde{\text{Conf}}(J) \times SU(2)$ there exists a one-to-one mapping between the quartic Killing potentials that generate the isometry group $\text{QConf}(J)$ under Poisson brackets in the HSS formulation, and the generators of the minimal unitary representation of $\text{QConf}(J)$. It would be important to understand physically how the minimal representation may arise by quantizing the sigma-model in harmonic superspace.

The main goal of the present work is to explain the relation between the twistorial construction of the quaternionic discrete series in [26] and the quasi-conformal actions discovered in [25], and moreover to elucidate the sense in which the minimal representation is obtained by “quantizing the quasi-conformal action”. While our results are unlikely to cause any surprise to the informed mathematician, we hope that our exegesis of [26] will be useful to physicists, *e.g.* in subsequent applications to supergravity and black holes, and possibly to mathematicians too, *e.g.* in obtaining explicit formulae for automorphic forms along the lines of [35].

As the main body of this paper is fairly technical, we summarize its content below, including some open problems and possible applications:

- (i) The key observation is that the variables (Ξ, α) of the quasi-conformal realization have a natural interpretation as *complex* coordinates on the twistor space

$\mathcal{Z} = G/U(1) \times M$ over the quaternionic-Kähler space $G/SU(2) \times M$, adapted to the action of the Heisenberg algebra in the nilpotent radical of the Heisenberg parabolic subgroup of G . The logarithm of the “quartic norm” N_4 provides a Kähler potential (2.57) for the G -invariant Einstein-Kähler metric on \mathcal{Z} .⁷ Using the Harish-Chandra decomposition, we also construct the complex coordinates adapted to another Heisenberg algebra related by a Cayley-type transform, whose center is a compact generator rather than a nilpotent one. These coordinates are the analogue of the Poincaré disk coordinates for $SL(2, \mathbb{R})/U(1)$, and it would be interesting to give a Jordan-type description of the corresponding Kähler potential, given in (2.39) for $G = SU(2, 1)$.

- (ii) It is also possible to view (Ξ, α) as *real* coordinates on G/P , where P is a parabolic subgroup of G with Heisenberg radical. This G/P arises as a piece of the boundary of \mathcal{Z} or of G/K . The “quartic light-cone” then provides the causal structure on the boundary, as shown in Section 2.4.6 for $G = SU(2, 1)$.
- (iii) The quasi-conformal action admits a continuous deformation by a parameter $k \in \mathbb{C}$, corresponding to the action of G on sections of a line bundle over G/P induced from a character of P . This action provides a degenerate principal series representation, which is manifestly unitary for $k \in 2 + i\mathbb{R}$ ($SU(2, 1)$) or $k \in 3 + i\mathbb{R}$ ($G_{2(2)}$). We tabulate the formulas for the infinitesimal action of \mathfrak{g} in this representation, and determine the spherical vector.
- (iv) When k is an integer, one can also consider sections on G/P which can be (in an appropriate sense) extended holomorphically into \mathcal{Z} . Thus, the quasi-conformal action with Ξ, α complex as in i) leads to a differential operator realization of the action of G on these sections. When $G = SU(2, 1)$ and $k \geq 3$, as explained in [26], this action gives quaternionic discrete series representations; we explicitly compute the K -finite vectors of this submodule in the principal series. When $G = G_{2(2)}$ we similarly study the K -finite vectors of the principal series and identify a natural subset which has the same K -type decomposition as the quaternionic discrete series. Moreover, in this case we study also the more intricate picture described in [26] for smaller values of k ; while the results there do not strictly apply to $G_{2(2)}$, we nevertheless find constraints defining “small” invariant submodules of the principal series at the expected values of k .

⁷This follows by specializing analysis performed in [36] of general “dual” quaternionic-Kähler spaces to symmetric spaces. It will be rederived below using purely group theoretic methods (in fact, this approach was a useful guide for the general analysis in [36]).

- (v) Given a class in $H^1(\mathcal{Z}, \mathcal{O}(-k))$, the Penrose transform produces a section of a vector bundle over the quaternionic-Kähler base G/K annihilated by some “quaternionic” differential operators [37, 38, 36]. For $G = SU(2, 1)$ and k even, we argue formally but explicitly in Section 2.4.5 that computing the Penrose transform of Ψ is equivalent to evaluating a matrix element between Ψ and the lowest K -type of the quaternionic discrete series. A similarly explicit understanding of the Penrose transform for k odd remains an open problem, which would require a proper prescription for dealing with the branch cuts in the formula of [36].
- (vi) The quasi-conformal action on \mathcal{Z} can be lifted to a tri-holomorphic action on the hyperkähler cone (or Swann space) $\mathcal{S} = \mathbb{R} \times G/M$, locally isomorphic to the smallest nilpotent coadjoint orbit of $G_{\mathbb{C}}$. In particular, the action of G on \mathcal{S} preserves the holomorphic symplectic form. The minimal representation of G can be viewed as the “holomorphic quantization” of \mathcal{S} . We show explicitly in Section 2.6 for $SU(2, 1)$ and 3.4.1 for $G_{2,2}$ that the leading differential symbols of the generators of the minimal representation are equal to the holomorphic moment maps of the action of G on \mathcal{S} , and identify the corresponding semi-classical limit.
- (vii) We determine explicitly the lowest K -type of the minimal representation, generalizing the analysis of [39] to these two quaternionic groups. For $G_{2(2)}$, the lowest K -type in the real polarization (3.130) bears strong similarities to the result found in [39] for simply laced split groups, while the wave function in the complex (upside-down) polarization (3.130) is analogous to the topological string wave function. (Such a relation is not unexpected, given the results of [15], which showed that the holomorphic anomaly equations of the topological string can be naturally explained in terms of the minimal representation.) We show further that the semi-classical limit of the lowest K -type wave function yields the generating function for a holomorphic Lagrangian cone inside the holomorphic symplectic space \mathcal{S} , invariant under the holomorphic action of G^8 . It would be very interesting to formulate the hyperkähler geometry of \mathcal{S} in terms of this Lagrangian cone, by analogy to special Kähler geometry.
- (viii) Deviating slightly from the main subject of this paper, we also give an explicit construction of a degenerate principal series of $G_{2(2)}$ induced from a parabolic subgroup P_3 (different from the Heisenberg parabolic), with nilpotent radical of order 3. We also describe a polarization of the minimal representation adapted to this parabolic subgroup. We expect that this construction will have applications

⁸This Lagrangian cone was already instrumental in [35], in extending the real spherical vector found in [39] to the adelic setting.

to the physics of supersymmetric black holes in 5 dimensions, along the lines discussed in [40, 41].

The organization of this paper is as follows: In Sections 2 and 3, for the two rank 2 quaternionic groups $G = SU(2, 1)$ and $G = G_{2(2)}$ successively, we give explicit parametrizations of the quaternionic-Kähler homogeneous spaces G/K and their twistor spaces, provide explicit differential operator realizations of the principal series, quaternionic discrete series and minimal representations, and compute their spherical or lowest K -type vectors.

Throughout this paper we work mostly at the level of the Lie algebras. We are not careful about the discrete factors in any of the various groups that appear. We are also not careful about the precise globalizations of the representations we consider (so we do not worry about whether we study L^2 functions, smooth functions, hyperfunctions etc); for the most part we are really considering only the (\mathfrak{g}, K) -modules consisting of the K -finite vectors. We emphasize concrete formulas even if they are formal, with the idea that these formal manipulations may be related to ones frequently carried out by physicists studying the topological string, in light of the relations between group representations and topological strings identified in [15].

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2. $SU(2, 1)$

2.1 Some group theory

The non-compact Lie group $G = SU(2, 1)$ is defined as the group of unimodular transformations of \mathbb{C}^3 which preserve a given hermitian metric η with signature $(+, +, -)$.

It is convenient to choose

$$\eta = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \quad (2.1)$$

The Lie algebra $\mathfrak{g} = su(2, 1)$ consists of traceless matrices such that $\eta X + X^\dagger \eta = 0$. This condition is solved by

$$X = \begin{pmatrix} \underline{H} + i\underline{J}/3 & \underline{E}_p - i\underline{E}_q & i\underline{E} \\ \underline{F}_p + i\underline{F}_q & -2i\underline{J}/3 & -(\underline{E}_p + i\underline{E}_q) \\ -i\underline{F} & -(\underline{F}_p - i\underline{F}_q) & -\underline{H} + i\underline{J}/3 \end{pmatrix} \equiv \underline{X}_i X_i \quad (2.2)$$

where $\{\underline{X}_i\} = \{\underline{H}, \underline{J}, \underline{E}_p, \underline{E}_q, \underline{E}, \underline{F}_p, \underline{F}_q, \underline{F}\}$ are the real coefficients of the generators $\{X_i\} = \{H, J, E_p, E_q, E, F_p, F_q, F\}$ in \mathfrak{g} . The latter are represented by anti-hermitean operators in any unitary representation. They obey the commutation relations (consistent with the matrix representation above)

$$\begin{aligned} [E_p, E] &= 0, & [E_q, E] &= 0, & [F_p, F] &= 0, & [F_q, F] &= 0, \\ [E_p, E_q] &= -2E, & [F_p, F_q] &= 2F, & [J, E_p] &= -E_q, & [J, F_p] &= -F_q, \\ [J, E_q] &= E_p, & [J, F_q] &= F_p, & [H, E_p] &= E_p, & [H, F_p] &= -F_p, \\ [H, E_q] &= E_q, & [H, F_q] &= -F_q, & [H, E] &= 2E, & [H, F] &= -2F, \\ [E_p, F_p] &= H, & [E_q, F_q] &= H, & [E_p, F_q] &= 3J, & [E_q, F_p] &= -3J, \\ [E, F_p] &= E_q, & [F, E_p] &= -F_q, & [E, F_q] &= -E_p, & [F, E_q] &= F_p, \\ [J, F] &= 0, & [J, E] &= 0, & [E, F] &= H, & [H, J] &= 0 \end{aligned} \quad (2.3)$$

The center of the universal enveloping algebra $U(\mathfrak{g})$ is generated by the quadratic Casimir

$$C_2 = \frac{1}{4}H^2 - \frac{3}{4}J^2 + \frac{1}{4}(E_p F_p + F_p E_p + E_q F_q + F_q E_q) + \frac{1}{2}(EF + FE) \quad (2.4)$$

and cubic Casimir

$$\begin{aligned} C_3 &= H^2 J + J^3 - E(F_p^2 + F_q^2) + F(E_p^2 + E_q^2) \\ &\quad + (F_q E_p - F_p E_q)H + (4E F - F_p E_p - F_q E_q - 2)J \end{aligned} \quad (2.5)$$

The Casimir operators take values [6]

$$C_2 = p + q + \frac{1}{3}(p^2 + pq + q^2), \quad C_3 = \frac{4i}{27}(p - q)(p + 2q + 3)(q + 2p + 3) \quad (2.6)$$

in a finite-dimensional representation of $SU(2, 1)$ corresponding to a tensor $T_{j_1 \dots j_q}^{i_1 \dots i_p}$ with p upper and q lower indices (in particular, $(C_2, C_3) = (3, 0)$ and $(4/3, 80i/27)$ for the

adjoint and fundamental representations, respectively). It is convenient to continue to use the same variables (p, q) defined in (2.6), no longer restricted to integers, to label the infinite-dimensional representations of $SU(2, 1)$. Alternatively, one may define the “infinitesimal character”

$$x_1 = -\frac{1}{3}(p + 2q + 3), \quad x_2 = \frac{1}{3}(p - q), \quad x_3 = \frac{1}{3}(2p + q + 3) \quad (2.7)$$

with $x_1 + x_2 + x_3 = 0$, such that the Weyl group of $SU(2, 1)$ acts by permutations of (x_1, x_2, x_3) . In a unitary representation, either all x_i are real, or one is real and the other two are complex conjugates [6]. In the former case one may choose to order $x_1 \leq x_2 \leq x_3$, corresponding to $p \geq -1, q \geq -1$. We shall be particularly interested in representations where $p = q$, such that the infinitesimal character is proportional to the Weyl vector $(-1, 0, 1)$.

The generators H, J generate a Cartan subalgebra of \mathfrak{g} . The spectrum of the adjoint action of

$$\text{Spec}(J) = \{0, \pm i\}, \quad \text{Spec}(H) = \{0, \pm 1, \pm 2\} \quad (2.8)$$

shows that H and J are non-compact and compact, respectively. In fact, the generator H gives rise to a “real non-compact” 5-grading

$$\mathfrak{g} = F|_{-2} \oplus \{F_p, F_q\}|_{-1} \oplus \{H, J\}|_0 \oplus \{E_p, E_q\}|_1 \oplus E|_2 \quad (2.9)$$

where the subscript denotes the eigenvalue under H . In this decomposition, each subspace is invariant under hermitian conjugation. Moreover $J \oplus \{E, H, F\}$ generate a $U(1) \times SL(2, \mathbb{R})$ (non-compact) maximal subgroup of G . The remaining roots arrange themselves into a pair of doublets of $SL(2, \mathbb{R})$ with opposite charge under J ,

$$\begin{pmatrix} F_p - iF_q & E_p - iE_q \\ F_p + iF_q & E_p + iE_q \end{pmatrix} \quad (2.10)$$

as shown on the root diagram 1. The parabolic subgroup $P = LN$ with Levi $L = \mathbb{R} \times U(1)$ generated by $\{H, J\}$ and unipotent radical N generated by $\{F_p, F_q, F\}$, corresponding to the spaces with zero and negative grade in the decomposition (3.12), is known as the Heisenberg parabolic subgroup, and will play a central rôle in all constructions in this paper.

For later purposes, it will be useful to introduce another basis of \mathfrak{g} adapted to a maximal *compact* subgroup $K = SU(2) \times U(1)$. We first go to a compact basis for the $SL(2, \mathbb{R})$ factor generated by E, F, H :

$$\begin{aligned} L_0 &= \frac{1}{2}(F - E), \quad K_0 = \frac{1}{4}(2L_0 + 3J), \quad L_{\pm} = -\frac{1}{2\sqrt{2}}(E + F \pm iH) \\ K_{\pm} &= -\frac{1}{4}[E_p \pm iE_q + (F_p \pm iF_q)], \quad J_{\pm} = -\frac{1}{2\sqrt{2}}[E_p \mp iE_q - (F_p \mp iF_q)] \end{aligned} \quad (2.11)$$

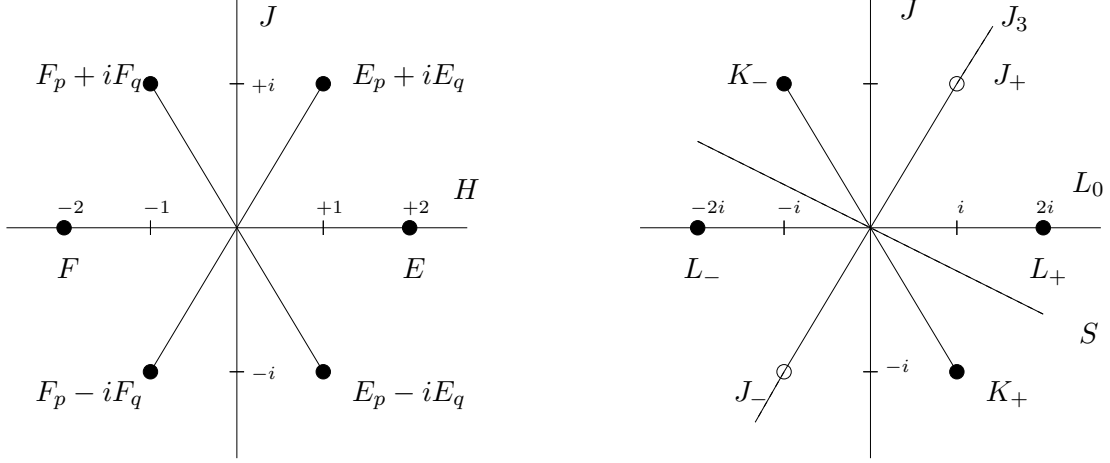


Figure 1: Root diagram of $SU(2,1)$ with respect to the mixed Cartan torus H, J (left) and the compact Cartan torus L_0, J (right). The compact (resp. non-compact) roots are indicated by a white (resp. black) dot.

Then $\{L_+, L_0, L_-\}$ and $\{K_+, K_0, K_-\}$ make two $SL(2, \mathbb{R})$ subalgebras, with compact Cartan generators L_0 and K_0 , respectively,

$$[L_0, L_{\pm}] = \pm i L_{\pm} , \quad [L_+, L_-] = -i L_0 \quad (2.12)$$

$$[K_0, K_{\pm}] = \pm i K_{\pm} , \quad [K_+, K_-] = -i K_0 \quad (2.13)$$

Since $\text{Spec}(L_0) = \{0, \pm \frac{i}{2}, \pm i\}$, (L_0, J) now form a Cartan torus, and L_0 gives rise to a new 5-grading

$$\mathfrak{g} = L_-|_{-i} \oplus \{K_-, J_-\}|_{-\frac{i}{2}} \oplus \{L_0, J\}|_0 \oplus \{J_+, K_+\}|_{\frac{i}{2}} \oplus L_+|_i \quad (2.14)$$

where the subscript denotes the eigenvalue under L_0 . Unlike the 5-grading (2.9), in (2.14) hermitean conjugation exchanges the positive and negative grade spaces.

Next, we perform a $\pi/3$ rotation of the root diagram, and define

$$J_3 = \frac{1}{4}(F - E - 3J) , \quad S = \frac{3}{4}(F - E + J) \quad (2.15)$$

$$J_{\pm \frac{1}{2}, \pm \frac{3}{2}} = \pm 2\sqrt{2}i L_{\pm} , \quad J_{\pm \frac{1}{2}, \mp \frac{3}{2}} = \sqrt{2} K_{\mp} \quad (2.16)$$

Then $\{J_+, J_3, J_-\}$ and S generate the compact subgroup $K = SU(2) \times U(1)$,⁹

$$[J_3, J_{\pm}] = \pm i J_{\pm} , \quad [J_+, J_-] = 2i J_3 \quad (2.17)$$

⁹To be precise, $K = (SU(2) \times U(1))/\mathbb{Z}_2$. We abuse notation by suppressing this \mathbb{Z}_2 in most of what follows.

In particular, J_3 induces the “compact 5-grading”

$$J_-|_{-i} \oplus \{L_-, K_+\}|_{-\frac{i}{2}} \oplus \{J_3, S\}|_0 \oplus \{K_-, L_+\}|_{\frac{i}{2}} \oplus J_+|_i \quad (2.18)$$

where the subscript now denotes the J_3 eigenvalue. The remaining roots can then be arranged as a pair of doublets under $SU(2)$ with opposite $U(1)_S$ charges,

$$\begin{pmatrix} J_{-\frac{1}{2}, -\frac{3}{2}} & J_{\frac{1}{2}, -\frac{3}{2}} \\ J_{-\frac{1}{2}, \frac{3}{2}} & J_{\frac{1}{2}, \frac{3}{2}} \end{pmatrix} \quad (2.19)$$

as shown on the second root diagram on Figure 1.

In order to represent the generators in the compact basis by pseudo-hermitian matrices, it is convenient to change basis and diagonalize the hermitian metric (2.1),

$$\eta' \equiv C\eta C^t = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \quad (2.20)$$

The generators preserving the metric η' can now be parametrized as

$$\begin{pmatrix} -\frac{i}{2}(\underline{J}_3 + \underline{S}) & 2\underline{J}_{-\frac{1}{2}, -\frac{3}{2}} & -\underline{J}_- \\ 2\underline{J}_{\frac{1}{2}, \frac{3}{2}} & i\underline{S} & -2\underline{J}_{-\frac{1}{2}, \frac{3}{2}} \\ \underline{J}_+ & -2\underline{J}_{\frac{1}{2}, -\frac{3}{2}} & \frac{i}{2}(\underline{J}_3 - \underline{S}) \end{pmatrix} \quad (2.21)$$

where $\underline{J}_3, \underline{S}$ are real, while $\underline{J}_\pm = \underline{J}_\mp^*$, $\underline{J}_{\pm\frac{1}{2}, \pm\frac{3}{2}} = \underline{J}_{\mp\frac{1}{2}, \mp\frac{3}{2}}^*$, $\underline{J}_{\pm\frac{1}{2}, \mp\frac{3}{2}} = \underline{J}_{\mp\frac{1}{2}, \pm\frac{3}{2}}^*$. By analogy with the $SL(2, \mathbb{R}) = SU(1, 1)$ case, we shall refer to the matrix C as a Cayley rotation.

2.2 Quaternionic symmetric space

We now describe the geometry of the quaternionic-Kähler symmetric space $K \backslash G = (SU(2) \times U(1)) \backslash SU(2, 1)$. This space is well known in the string theory literature as the tree-level moduli space of the universal hypermultiplet (see e.g. [11, 42, 43, 44] for some useful background). It is in the class of “dual quaternionic manifolds”, in the sense that it can be constructed by the c -map procedure [45, 12] from the trivial zero-dimensional special Kähler manifold with quadratic prepotential $F = -i(X^0)^2/2$.

In order to parameterize this space, we use the Iwasawa decomposition of G

$$g = k \cdot e^{-UH} \cdot e^{\tilde{\zeta}E_p - \zeta E_q} \cdot e^{\sigma E} \quad (2.22)$$

where k is an element of the maximal compact subgroup $K = SU(2) \times U(1)$. In terms of the fundamental representation, this is $g = k \cdot e_{QK}$ where e_{QK} is the coset representative

$$e_{QK} = \begin{pmatrix} e^{-U} & & \\ & 1 & \\ & & e^U \end{pmatrix} \cdot \begin{pmatrix} 1 & \tilde{\zeta} + i\zeta & i\sigma - \frac{1}{2}(\tilde{\zeta}^2 + \zeta^2) \\ & 1 & -(\tilde{\zeta} - i\zeta) \\ & & 1 \end{pmatrix}, \quad (2.23)$$

The right-invariant \mathfrak{g} -valued 1-form is then

$$\theta = de_{QK} e_{QK}^{-1} = \begin{pmatrix} -dU & e^{-U}(d\tilde{\zeta} + id\zeta) & ie^{-2U}(d\sigma + \tilde{\zeta}d\zeta - \zeta d\tilde{\zeta}) \\ 0 & 0 & -e^{-U}(d\tilde{\zeta} - id\zeta) \\ 0 & 0 & dU \end{pmatrix} \quad (2.24)$$

Expanding θ on the compact basis of \mathfrak{g} , its 1-form components are

$$\begin{pmatrix} \underline{J}_{-\frac{1}{2}, \frac{3}{2}} & \underline{J}_{\frac{1}{2}, \frac{3}{2}} \\ \underline{J}_{-\frac{1}{2}, -\frac{3}{2}} & \underline{J}_{\frac{1}{2}, -\frac{3}{2}} \end{pmatrix} = - \begin{pmatrix} \bar{u} & v \\ \bar{v} & u \end{pmatrix} , \quad (2.25)$$

$$\begin{pmatrix} \underline{J}_- \\ \underline{J}_3 \\ \underline{J}_+ \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \bar{u} \\ \frac{1}{4i}(v - \bar{v}) \\ u \end{pmatrix} , \quad \underline{S} = \frac{3i}{8}(v - \bar{v}) \quad (2.26)$$

where we defined the 1-forms

$$u = -\frac{1}{\sqrt{2}}e^{-U}(d\tilde{\zeta} + id\zeta) , \quad v = dU + \frac{i}{2}e^{-2U}(d\sigma + \tilde{\zeta}d\zeta - \zeta d\tilde{\zeta}) \quad (2.27)$$

The non-compact components (2.25) give the quaternionic viel-bein of the invariant metric, while the compact components (2.26) give the spin connection, with restricted holonomy $SU(2) \times U(1)$. The invariant metric on $K \backslash G$ is thus given by

$$\begin{aligned} ds^2 &= 2(u\bar{u} + v\bar{v}) \\ &= 2(dU)^2 + e^{-2U} \left(d\tilde{\zeta}^2 + d\zeta^2 \right) + \frac{1}{2}e^{-4U} \left(d\sigma + \tilde{\zeta}d\zeta - \zeta d\tilde{\zeta} \right)^2 \end{aligned} \quad (2.28)$$

An exception among quaternionic-Kähler, this metric is Kähler in the complex structure induced from the $U(1)$ generator S . A Kähler potential is given by $-\log(s + \bar{s} - 2c\bar{c})$, where s, c are the complex coordinates

$$s = e^{2U} + \zeta^2 + \tilde{\zeta}^2 + i\sigma , \quad c = \tilde{\zeta} + i\zeta . \quad (2.29)$$

The group G acts on the coset space $K \backslash G$ by right multiplication, followed by a left action of the maximal compact $K = SU(2) \times U(1)$ so as to maintain the Iwasawa gauge $k = 1$ in (2.22). The metric is invariant under this action. This gives an action

of $SU(2, 1)$ by Killing vectors on $K \backslash G$:

$$E^{QK} = \partial_\sigma, \quad E_p^{QK} = \partial_{\tilde{\zeta}} - \zeta \partial_\sigma, \quad E_q^{QK} = -\partial_\zeta - \tilde{\zeta} \partial_\sigma, \quad (2.30a)$$

$$J^{QK} = \zeta \partial_{\tilde{\zeta}} - \tilde{\zeta} \partial_\zeta, \quad H^{QK} = -\partial_U - \zeta \partial_\zeta - \tilde{\zeta} \partial_{\tilde{\zeta}} - 2\sigma \partial_\sigma \quad (2.30b)$$

$$\begin{aligned} F_p^{QK} = & -\tilde{\zeta} \partial_U - (\sigma + 2\zeta \tilde{\zeta}) \partial_\zeta + \left[e^{2U} + \frac{1}{2}(3\zeta^2 - \tilde{\zeta}^2) \right] \partial_{\tilde{\zeta}} \\ & + \left[\zeta \left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2) \right) - \sigma \tilde{\zeta} \right] \partial_\sigma \end{aligned} \quad (2.30c)$$

$$\begin{aligned} F_q^{QK} = & \zeta \partial_U - \left[e^{2U} + \frac{1}{2}(3\tilde{\zeta}^2 - \zeta^2) \right] \partial_{\tilde{\zeta}} - (\sigma - 2\zeta \tilde{\zeta}) \partial_\zeta \\ & + \left[\tilde{\zeta} \left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2) \right) + \sigma \zeta \right] \partial_\sigma \end{aligned} \quad (2.30d)$$

$$\begin{aligned} F^{QK} = & -\sigma \partial_U + \left[\left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2) \right)^2 - \sigma^2 \right] \partial_\sigma - \left[\tilde{\zeta} \left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2) \right) + \sigma \zeta \right] \partial_\zeta \\ & + \left[\zeta \left(e^{2U} + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2) \right) - \sigma \tilde{\zeta} \right] \partial_{\tilde{\zeta}} \end{aligned} \quad (2.30e)$$

The action of G on $K \backslash G$ also induces a representation of G on $L^2(K \backslash G)$. The quadratic Casimir in this representation is proportional to the Laplace-Beltrami operator of the metric (2.28), while the cubic Casimir vanishes identically:

$$C_2 = \frac{1}{4} \Delta = \frac{1}{4} \frac{1}{\sqrt{g}} \partial_i \sqrt{g} g^{ij} \partial_j, \quad C_3 = 0 \quad (2.31)$$

2.3 Twistor space and Swann space

The twistor space \mathcal{Z} is a $\mathbb{CP}^1 = U(1) \backslash SU(2)$ bundle over the quaternionic-Kähler space $(SU(2) \times U(1)) \backslash SU(2, 1)$, which carries a Kähler-Einstein metric. The fibration is such that the $SU(2)$ “cancels” [46], so that \mathcal{Z} is an homogeneous (but not symmetric) space,

$$\mathcal{Z} = (U(1)_S \times U(1)_{J_3}) \backslash SU(2, 1) \quad (2.32)$$

Let H denote the subgroup $U(1)_S \times U(1)_{J_3}$. In the following we construct two canonical sets of complex coordinates on \mathcal{Z} , adapted to two different Heisenberg algebras, and relate them to the coordinates $U, \zeta, \tilde{\zeta}, \sigma$ on the base and the stereographic coordinate z on the sphere.¹⁰

¹⁰Of course, these “coordinates on \mathcal{Z} ” really cover only an open dense subset; they become singular at the (canonically defined) north and south poles of the \mathbb{CP}^1 fibers.

2.3.1 Harish-Chandra coordinates

The complex structure on \mathcal{Z} can be constructed by using the Borel embedding $H \backslash G \hookrightarrow P'_\mathbb{C} \backslash G_\mathbb{C}$ where $P'_\mathbb{C}$ is the parabolic (Borel) subgroup of the complexified group $G_\mathbb{C}$, generated by the positive roots $J_{\frac{1}{2}, \pm \frac{3}{2}}, J_+$ and Cartan generators J_3, S . To obtain complex coordinates from this embedding we go to the Cayley-rotated matrix representation (2.21), and perform a $\bar{N}' A' N'$ decomposition,

$$C e_{\mathcal{Z}} C^{-1} = \begin{pmatrix} 1 & & \\ \tilde{p} & 1 & \\ \tilde{k} & \tilde{q} & 1 \end{pmatrix} \cdot \begin{pmatrix} ab & & \\ & 1/a^2 & \\ & & a/b \end{pmatrix} \cdot \begin{pmatrix} 1 & p & k \\ & 1 & q \\ & & 1 \end{pmatrix} \quad (2.33)$$

The entries p, q, k in the upper-triangular matrix by construction provide holomorphic coordinates on \mathcal{Z} . The lower-triangular and diagonal matrices are then expressed in terms of p, q, k and their complex conjugates $\bar{p}, \bar{q}, \bar{k}$ by requiring that (2.33) is an element of $SU(2, 1)$ rather than of its complexification:

$$\tilde{p} = \frac{\bar{p} - \bar{k}q}{\sqrt{\Sigma}} \ , \quad \tilde{q} = \frac{\bar{k}p - \bar{p}\bar{q}p + \bar{q}}{\sqrt{k\bar{k} - p\bar{p} + 1}} \ , \quad \tilde{k} = \frac{(\bar{p}\bar{q} - \bar{k})\sqrt{k\bar{k} - p\bar{p} + 1}}{\sqrt{\Sigma}} \quad (2.34a)$$

$$a = \frac{\Sigma^{1/4}}{(k\bar{k} - p\bar{p} + 1)^{1/4}} \ , \quad b = \frac{1}{(k\bar{k} - p\bar{p} + 1)^{1/4} \Sigma^{1/4}} \quad (2.34b)$$

where

$$\Sigma = 1 + k\bar{k} - q\bar{q} - \bar{k}p q - k\bar{p}\bar{q} + p\bar{p}q\bar{q} \quad (2.35)$$

These coordinates are adapted to the holomorphic action of the Heisenberg algebra generated by $J_{-\frac{1}{2}, \pm \frac{3}{2}}, J_-$, in the sense that

$$J_{-\frac{1}{2}, \frac{3}{2}} = -2(\partial_q + p\partial_k) \ , \quad J_{-\frac{1}{2}, -\frac{3}{2}} = 2\partial_p \ , \quad J_- = -\partial_k \quad (2.36)$$

It is useful to record the action of the other generators in the compact basis,

$$J_3 = \frac{i}{2}(p\partial_p + q\partial_q + 2k\partial_k) \ , \quad S = \frac{3i}{2}(p\partial_p - q\partial_q) \ , \quad (2.37a)$$

$$J_+ = -kp\partial_p - (k - pq)q\partial_q - k^2\partial_k \quad (2.37b)$$

$$J_{\frac{1}{2}, \frac{3}{2}} = -2p^2\partial_p - 2(k - pq)\partial_q - 2kp\partial_k \ , \quad J_{\frac{1}{2}, -\frac{3}{2}} = -2k\partial_p + 2q^2\partial_q \quad (2.37c)$$

A G -invariant metric on \mathcal{Z} can be constructed in the usual way, by applying an L -invariant quadratic form on \mathfrak{g} to the \mathfrak{g} -valued 1-form $\theta = de_{\mathcal{Z}} \cdot e_{\mathcal{Z}}^{-1}$. In contrast to the case of $K \backslash G$, this quadratic form is not unique up to scalar multiple, but has parameters $(\alpha, \beta, \gamma) \in \mathbb{R}^3$:

$$\alpha \underline{J}_+ \underline{J}_- + \beta \underline{J}_{\frac{1}{2}, \frac{3}{2}} \underline{J}_{-\frac{1}{2}, -\frac{3}{2}} + \gamma \underline{J}_{\frac{1}{2}, -\frac{3}{2}} \underline{J}_{-\frac{1}{2}, \frac{3}{2}} \quad (2.38)$$

These parameters can be fixed by requiring that the resulting metric on \mathcal{Z} is Einstein-Kähler. In particular, the Kähler potential must be proportional to the volume element. This uniquely fixes $\alpha = -2, \beta = \gamma = 4$ (up to rescalings) and gives the Kähler potential

$$K_{\mathcal{Z}} = \frac{1}{2} \log [(1 + k\bar{k} - p\bar{p})(1 + |k - pq|^2 - q\bar{q})] \quad (2.39)$$

This reproduces Eq. 7.28 in [47] upon identifying $k = \zeta/2, p = v, q = -u$. Under the action of $J_{\frac{1}{2}, \frac{3}{2}} + J_{-\frac{1}{2}, -\frac{3}{2}}, J_{\frac{1}{2}, -\frac{3}{2}} + J_{-\frac{1}{2}, \frac{3}{2}}, J_+ + J_-$, $K_{\mathcal{Z}}$ transforms by a Kähler transformation $K_{\mathcal{Z}} \rightarrow K_{\mathcal{Z}} + f + \bar{f}$, with f proportional to $p, q, k - \frac{1}{2}pq$, respectively.

2.3.2 Iwasawa coordinates

We now exhibit the twistor space \mathcal{Z} as an S^2 fibration over $K \backslash G$, such that the S^2 fiber over any point on the base is holomorphically embedded in \mathcal{Z} . For this purpose, we recall that the complex structure on $S^2 = U(1) \backslash SU(2)$ can be constructed using the Borel embedding $U(1) \backslash SU(2) \hookrightarrow B_{\mathbb{C}} \backslash SL(2, \mathbb{C})$. Here $SL(2, \mathbb{C}) \subset G_{\mathbb{C}}$ is generated by J_+, J_3, J_- and $B_{\mathbb{C}}$ is the Borel subgroup generated by J_+, J_3 . The embedding is simply obtained by starting with a coset representative $e \in U(1) \backslash SU(2)$ and viewing it instead as a representative in $B_{\mathbb{C}} \backslash SL(2, \mathbb{C})$ (this is consistent since $B_{\mathbb{C}} \cap SU(2) = U(1)$.) The same class in $B_{\mathbb{C}} \backslash SL(2, \mathbb{C})$ is also represented by $\exp z J_-$ for some $z \in \mathbb{C}$ (with one exception corresponding to $z = \infty$), giving the desired complex coordinate.

Now we use the same idea for \mathcal{Z} . So we return to the original matrix representation (2.2), and parameterize \mathcal{Z} by the coset representative

$$e_{\mathcal{Z}} = e^{-\bar{z}J_+} (1 + z\bar{z})^{-iJ_3} e^{-zJ_-} e_{QK} \quad (2.40)$$

$$= \frac{1}{\sqrt{1 + z\bar{z}}} \begin{pmatrix} \frac{1}{2}(1 + \sqrt{1 + z\bar{z}}) & \frac{z}{\sqrt{2}} & \frac{1}{2}(1 - \sqrt{1 + z\bar{z}}) \\ -\frac{1}{\sqrt{2}}\bar{z} & 1 & -\frac{1}{\sqrt{2}}\bar{z} \\ \frac{1}{2}(1 - \sqrt{1 + z\bar{z}}) & \frac{z}{\sqrt{2}} & \frac{1}{2}(1 + \sqrt{1 + z\bar{z}}) \end{pmatrix} \cdot e_{QK} \quad (2.41)$$

where e_{QK} is a representative for $K \backslash G$ in the Iwasawa decomposition (2.23). The coordinate z is then a stereographic coordinate on the S^2 fiber over each point of $K \backslash G$.

By Cayley rotating (2.40) and performing the Harish-Chandra decomposition (2.33), we can now relate the complex coordinates p, q, k and their complex conjugates to the

coordinates $U, \zeta, \tilde{\zeta}, \sigma$ on the base and the coordinate z on the fiber:

$$p = \frac{4}{2 + 2i\sigma + 2e^{2U} - \zeta^2 - \tilde{\zeta}^2 + 2i\sqrt{2}e^U z(\zeta + i\tilde{\zeta})} - 1 \quad (2.42a)$$

$$q = \frac{2\sqrt{2}e^U(i\zeta + \tilde{\zeta}) - (\zeta^2 + \tilde{\zeta}^2 + 2i\sigma + 2 - 2e^{2U})z}{2i\sqrt{2}z(\zeta + i\tilde{\zeta}) + 4e^U} \quad (2.42b)$$

$$k = \frac{2(2e^U z + \sqrt{2}(i\zeta + \tilde{\zeta}))}{2 + 2i\sigma + 2e^{2U} - \zeta^2 - \tilde{\zeta}^2 + 2i\sqrt{2}e^U z(\zeta + i\tilde{\zeta})} \quad (2.42c)$$

Rather than obtaining the metric on \mathcal{Z} in the coordinates $U, \zeta, \tilde{\zeta}, \sigma, z, \bar{z}$ from the Kähler potential (2.39) by following the change of variables, we can simply decompose the invariant form $\theta_{\mathcal{Z}} = de_{\mathcal{Z}}e_{\mathcal{Z}}^{-1}$ and plug into (2.38) using the values of α, β, γ that were determined above. The components of $\theta_{\mathcal{Z}}$ along $(U(1) \times U(1)) \backslash SU(2, 1)$ read

$$\begin{pmatrix} \underline{J}_{-\frac{1}{2}, \frac{3}{2}} & \underline{J}_{\frac{1}{2}, \frac{3}{2}} \\ \underline{J}_{-\frac{1}{2}, -\frac{3}{2}} & \underline{J}_{\frac{1}{2}, -\frac{3}{2}} \end{pmatrix} = -\frac{1}{\sqrt{1+z\bar{z}}} \begin{pmatrix} \bar{u} + \bar{z}v & v - z\bar{u} \\ \bar{v} - \bar{z}u & u + z\bar{v} \end{pmatrix} \quad (2.43a)$$

$$\underline{J}_+ = -\frac{1}{1+z\bar{z}} \left(d\bar{z} + \bar{u} + \frac{1}{2}\bar{z}(v - \bar{v}) + \bar{z}^2 u \right) \equiv -\frac{D\bar{z}}{1+z\bar{z}} \quad (2.43b)$$

$$\underline{J}_- = -\frac{1}{1+z\bar{z}} \left(dz + u - \frac{1}{2}z(v - \bar{v}) + z^2 \bar{u} \right) \equiv -\frac{Dz}{1+z\bar{z}} \quad (2.43c)$$

leading to the Kähler-Einstein metric

$$ds_{\mathcal{Z}}^2 = u\bar{u} + v\bar{v} - 2\frac{DzD\bar{z}}{(1+z\bar{z})^2} \quad (2.44)$$

with signature $(4, 2)$. The connection term in Dz is recognized as the projective $SU(2)$ connection,

$$Dz = dz - \frac{1}{2}(A_1 + iA_2) + iA_3z - \frac{1}{2}(A_1 - iA_2)z^2 \quad (2.45)$$

$$= dz + u - \frac{1}{2}z(v - \bar{v}) + z^2 \bar{u} \quad (2.46)$$

where

$$A_1 = -(u + \bar{u}), \quad A_2 = i(u - \bar{u}), \quad A_3 = \frac{i}{2}(v - \bar{v}) \quad (2.47)$$

are the components of the $SU(2)$ spin connection computed in (2.26). A basis of holomorphic $(1, 0)$ forms providing a holomorphic viel-bein of \mathcal{Z} is given by the components of $\theta_{\mathcal{Z}}$ with negative weight under J_3 ,

$$\mathcal{V} = \frac{1}{\sqrt{1+z\bar{z}}} \left(\sqrt{2}Dz, u + z\bar{v}, v - z\bar{u} \right) \quad (2.48)$$

The Kähler form can be written as

$$\omega_{\mathcal{Z}} = -\frac{Dz D\bar{z}}{(1+z\bar{z})^2} + i x_a \omega^a \quad (2.49)$$

where x^a is the unit length vector with stereographic coordinate z ,

$$x_1 = \frac{z + \bar{z}}{1 + z\bar{z}}, \quad x_2 = \frac{i(z - \bar{z})}{1 + z\bar{z}}, \quad x_3 = \frac{1 - z\bar{z}}{1 + z\bar{z}} \quad (2.50)$$

and ω_a are the quaternionic 2-forms on the base,

$$\omega^1 = \frac{1}{2i}(uv - \bar{u}\bar{v}), \quad \omega^2 = \frac{1}{2}(uv + \bar{u}\bar{v}), \quad \omega^3 = \frac{1}{2i}(u\bar{v} + v\bar{u}) \quad (2.51)$$

It may be checked that this triplet of 2-forms satisfies the constraints from quaternionic-Kähler geometry,

$$d\omega^i + \epsilon_{ijk} A_j \wedge \omega^k = 0, \quad dA_i + \epsilon_{ijk} A_j \wedge A_k = 2\omega_i \quad (2.52)$$

2.3.3 Complex c -map coordinates

While the complex coordinates p, q, k are adapted to the action of the generators $J_{-\frac{1}{2}, \frac{3}{2}}, J_{-\frac{1}{2}, -\frac{3}{2}}, J_-$, in the sequel it will be useful to have complex coordinates $\xi, \tilde{\xi}, \alpha$ adapted to the “non-compact” Heisenberg algebra E_p, E_q, E , i.e. such that the action of these generators takes the canonical form

$$E_p = \partial_{\tilde{\xi}} + \xi \partial_{\alpha}, \quad E_q = -\partial_{\xi} + \tilde{\xi} \partial_{\alpha}, \quad E = -\partial_{\alpha} \quad (2.53)$$

(Since G acts holomorphically, we have abused notation by writing only the holomorphic part; the real vector fields would be obtained by adding the complex conjugates, in (2.53) and below.)

The change of variables can be found by diagonalizing the action of these generators in the p, q, k variables, leading to

$$\xi = \frac{i(k^2 - (p+1)qk + p+1)}{\sqrt{2}(p+1)(-k + pq + q)}, \quad \tilde{\xi} = \frac{-k^2 + (p+1)qk + p+1}{\sqrt{2}(p+1)(-k + pq + q)} \quad (2.54a)$$

$$\alpha = \frac{i(q + kp - p^2q)}{(p+1)(k - pq - q)} \quad (2.54b)$$

or, conversely,

$$p = -\frac{\xi^2 + \tilde{\xi}^2 - 2i\alpha + 2}{\xi^2 + \tilde{\xi}^2 - 2i\alpha - 2}, \quad q = \frac{i(\xi^2 + \tilde{\xi}^2 + 2i\alpha + 2)}{2\sqrt{2}(\xi + i\tilde{\xi})}, \quad k = -\frac{2i\sqrt{2}(\xi - i\tilde{\xi})}{\xi^2 + \tilde{\xi}^2 - 2i\alpha - 2} \quad (2.55)$$

The full action of G is then given by

$$E^{QC} = -\partial_\alpha, \quad E_p^{QC} = \partial_{\tilde{\xi}} + \xi \partial_\alpha, \quad E_q^{QC} = -\partial_\xi + \tilde{\xi} \partial_\alpha, \quad (2.56a)$$

$$H^{QC} = -\tilde{\xi} \partial_{\tilde{\xi}} - \xi \partial_\xi - 2\alpha \partial_\alpha, \quad J^{QC} = -\tilde{\xi} \partial_\xi + \xi \partial_{\tilde{\xi}} \quad (2.56b)$$

$$F_p^{QC} = \frac{1}{2}(3\xi^2 - \tilde{\xi}^2) \partial_{\tilde{\xi}} + (\alpha - 2\tilde{\xi}\xi) \partial_\xi - \frac{1}{2} \left[\xi(\tilde{\xi}^2 + \xi^2) + 2\alpha\tilde{\xi} \right] \partial_\alpha \quad (2.56c)$$

$$F_q^{QC} = -\frac{1}{2}(3\tilde{\xi}^2 - \xi^2) \partial_\xi + (\alpha + 2\tilde{\xi}\xi) \partial_{\tilde{\xi}} - \frac{1}{2} \left[\tilde{\xi}(\xi^2 + \tilde{\xi}^2) - 2\alpha\xi \right] \partial_\alpha \quad (2.56d)$$

$$F^{QC} = \frac{1}{2} \left[\xi(\tilde{\xi}^2 + \xi^2) + 2\alpha\tilde{\xi} \right] \partial_{\tilde{\xi}} - \frac{1}{2} \left[\tilde{\xi}(\xi^2 + \tilde{\xi}^2) - 2\alpha\xi \right] \partial_\xi \\ - \frac{1}{4} \left[(\tilde{\xi}^2 + \xi^2)^2 - 4\alpha^2 \right] \partial_\alpha \quad (2.56e)$$

In the new coordinate system $\xi, \tilde{\xi}, \alpha$, the Kähler potential (2.39) (after a Kähler transformation by $f = (p+1)(k-q-pq)$) is

$$K_{\mathcal{Z}} = \frac{1}{2} \log N_4 = \frac{1}{2} \log \left(\left[(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2 \right]^2 + 4(\alpha - \bar{\alpha} + \xi\bar{\tilde{\xi}} - \tilde{\xi}\bar{\xi})^2 \right) \quad (2.57)$$

Here we note that $N_4(\xi, \tilde{\xi}, \alpha; \bar{\xi}, \bar{\tilde{\xi}}, \bar{\alpha})$ is the quartic distance function of quasi-conformal geometry. Since G acts by isometries on \mathcal{Z} , it leaves the Kähler potential (2.57) invariant up to Kähler transformations. Equivalently, the quartic norm $N_4(\xi, \tilde{\xi}, \alpha; \bar{\xi}, \bar{\tilde{\xi}}, \bar{\alpha})$ defined by (2.57) transforms multiplicatively by a factor $f(\xi, \tilde{\xi}, \alpha) \bar{f}(\bar{\xi}, \bar{\tilde{\xi}}, \bar{\alpha})$, where the holomorphic function f depends on the generator under consideration. In particular, the “quartic light-cone” $N_4 = 0$ is invariant under the full action of $SU(2, 1)$, which motivated the appellation “quasi-conformal action” in [25].

Such a coordinate system adapted to the holomorphic action of a Heisenberg group exists for any c -map space and was used heavily in [36]. Moreover, the result (2.57) agrees with a general formula for the Kähler potential given there.

2.3.4 Twistor map

Combining the changes of variables (2.42a) and (2.54), we find that the complex coordinates $(\xi, \tilde{\xi}, \alpha)$ are expressed in terms of the Iwasawa coordinates $U, \zeta, \tilde{\zeta}, \sigma, z, \bar{z}$ as

$$\xi = \zeta - \frac{i}{\sqrt{2}} e^U (z + z^{-1}) \quad (2.58a)$$

$$\tilde{\xi} = \tilde{\zeta} + \frac{1}{\sqrt{2}} e^U (z - z^{-1}) \quad (2.58b)$$

$$\alpha = \sigma + \frac{1}{\sqrt{2}} e^U \left[z(\zeta + i\tilde{\zeta}) + z^{-1}(-\zeta + i\tilde{\zeta}) \right] \quad (2.58c)$$

Again $(\xi, \tilde{\xi}, \alpha)$ is a holomorphic (rational) function of z , at fixed values of $U, \zeta, \tilde{\zeta}, \sigma$, so that the twistor fiber over any point is a rational curve in \mathcal{Z} . Conversely,

$$e^{2U} = -\frac{N_4}{8[(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2]} \quad (2.59a)$$

$$\zeta = \frac{2(\alpha - \bar{\alpha})(\tilde{\xi} - \bar{\tilde{\xi}}) + (\xi - \bar{\xi}) (\xi^2 - \bar{\xi}^2 + \tilde{\xi}^2 - \bar{\tilde{\xi}}^2)}{2[(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2]} \quad (2.59b)$$

$$\tilde{\zeta} = \frac{2(\bar{\alpha} - \alpha)(\xi - \bar{\xi}) + (\tilde{\xi} - \bar{\tilde{\xi}}) (\xi^2 - \bar{\xi}^2 + \tilde{\xi}^2 - \bar{\tilde{\xi}}^2)}{2[(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2]} \quad (2.59c)$$

$$\sigma = \frac{\alpha + \bar{\alpha}}{2} - \frac{(\alpha - \bar{\alpha} + \xi\tilde{\xi} - \bar{\xi}\bar{\tilde{\xi}}) (\xi^2 - \bar{\xi}^2 + \tilde{\xi}^2 - \bar{\tilde{\xi}}^2)}{2[(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2]} \quad (2.59d)$$

$$z = \sqrt{\frac{\xi - \bar{\xi} - i(\tilde{\xi} - \bar{\tilde{\xi}})}{\xi - \bar{\xi} + i(\tilde{\xi} - \bar{\tilde{\xi}})}} \times \sqrt{\frac{(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2 - 2i(\alpha - \bar{\alpha} + \xi\tilde{\xi} - \bar{\xi}\bar{\tilde{\xi}})}{(\xi - \bar{\xi})^2 + (\tilde{\xi} - \bar{\tilde{\xi}})^2 + 2i(\alpha - \bar{\alpha} + \xi\tilde{\xi} - \bar{\xi}\bar{\tilde{\xi}})}} \quad (2.59e)$$

These formulae are in agreement with the general results in [36].

Using the twistor map (2.58), one finds that the action of G on \mathcal{Z} reproduces the action on the base (2.30a), plus an action along the fiber:

$$\begin{aligned} E^{QC} + \bar{E}^{QC} &= E^{QK}, & E_p^{QC} + \bar{E}_p^{QC} &= E_p^{QK}, & E_q^{QC} + \bar{E}_q^{QC} &= E_q^{QK}, \\ H^{QC} + \bar{H}^{QC} &= H^{QK}, & J^{QC} + \bar{J}^{QC} &= J^{QK} - i(z\partial_z - \bar{z}\partial_{\bar{z}}) \\ F_p^{QC} + \bar{F}_p^{QC} &= F_p^{QK} - \sqrt{2}e^U [(1+z^2)\partial_z + (1+\bar{z}^2)\partial_{\bar{z}}] - 3i\zeta(z\partial_z - \bar{z}\partial_{\bar{z}}) \\ F_q^{QC} + \bar{F}_q^{QC} &= F_q^{QK} + i\sqrt{2}e^U [(1-z^2)\partial_z - (1-\bar{z}^2)\partial_{\bar{z}}] - 3i\tilde{\zeta}(z\partial_z - \bar{z}\partial_{\bar{z}}) \\ F^{QC} + \bar{F}^{QC} &= F^{QK} + \left(ie^{2U} - 3(\zeta^2 + \tilde{\zeta}^2)\right) (z\partial_z - \bar{z}\partial_{\bar{z}}) \\ &\quad - \sqrt{2}e^U \left[\left((1+z^2)\zeta - i(1-z^2)\tilde{\zeta}\right) \partial_z + \left((1+\bar{z}^2)\zeta + i(1-\bar{z}^2)\tilde{\zeta}\right) \partial_{\bar{z}} \right] \end{aligned} \quad (2.60)$$

These formulae will be useful in Section 2.4.5 when we discuss the Penrose transform.

At this stage, we note that the twistor map relations (2.58) can be obtained more directly by the following trick. Recall that P is the parabolic subgroup of lower-triangular matrices in the matrix representation (2.2), and let \bar{P} be its opposite subgroup consisting of upper-triangular matrices. Then we construct a new embedding $H \backslash G \hookrightarrow P_{\mathbb{C}} \backslash G_{\mathbb{C}}$ by choosing a different coset representative: we use the formula (2.40) for $e_{\mathcal{Z}}$ but make an analytic continuation, regarding z and \bar{z} as independent and then taking a limit

$$\bar{z} \rightarrow -1/z \quad (2.61)$$

If we ignore for a moment the fact that the limit (2.61) of (2.40) is singular, formally it defines an element of $G_{\mathbb{C}}$. Moreover, the locus of elements in $G_{\mathbb{C}}$ so obtained is formally invariant under G , essentially because (2.61) is constructed from the antipodal map (real structure) on \mathbb{CP}^1 , which is $SU(2)$ -invariant. This locus will give the desired copy of $H \backslash G$ inside $P_{\mathbb{C}} \backslash G_{\mathbb{C}}$. We now define the coordinates $(\xi, \tilde{\xi}, \alpha)$ in $P_{\mathbb{C}} \backslash G_{\mathbb{C}}$ using the $N_{\mathbb{C}} A \bar{N}_{\mathbb{C}}$ decomposition,

$$e_{\mathcal{Z}} = \begin{pmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{pmatrix} \cdot \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \cdot \begin{pmatrix} 1 & \tilde{\xi} + i\xi & i\alpha - \frac{1}{2}(\xi^2 + \tilde{\xi}^2) \\ & 1 & -\tilde{\xi} + i\xi \\ & & 1 \end{pmatrix} \quad (2.62)$$

Then a direct matrix computation gives the coordinates of $e_{\mathcal{Z}}$ as

$$\xi = \zeta - \frac{i}{\sqrt{2}} \left[\frac{-1 + \sqrt{1 + z\bar{z}}}{z\bar{z}} \right] e^U (z - \bar{z}) \quad (2.63a)$$

$$\tilde{\xi} = \tilde{\zeta} + \frac{1}{\sqrt{2}} \left[\frac{-1 + \sqrt{1 + z\bar{z}}}{z\bar{z}} \right] e^U (z + \bar{z}) \quad (2.63b)$$

$$\alpha = \sigma + \frac{1}{\sqrt{2}} \left[\frac{-1 + \sqrt{1 + z\bar{z}}}{z\bar{z}} \right] e^U \left[z(\zeta + i\tilde{\zeta}) - \bar{z}(-\zeta + i\tilde{\zeta}) \right] \quad (2.63c)$$

recovering the desired result (2.58) after setting $\bar{z} = -1/z$. The result shows that the limit (2.61) is in fact regular in $N_{\mathbb{C}} A \backslash G_{\mathbb{C}}$ (although not in $G_{\mathbb{C}}$).

2.3.5 Swann space

There is an important complex line bundle over \mathcal{Z} , which we call $\mathcal{O}(2)$. It may be defined in two different ways. One way is to use $\mathcal{Z} = H \backslash G$. Then $\mathcal{O}(2)$ is determined by the character $\exp i\underline{J}_3$ of H — in other words, a section of $\mathcal{O}(2)$ is a function on G which transforms under H by this character.¹¹ This definition makes it easy to prove that $\mathcal{O}(2)$ admits a Hermitian structure.

On the other hand, one can also define $\mathcal{O}(2)$ using the Borel embedding $\mathcal{Z} \hookrightarrow P_{\mathbb{C}} \backslash G_{\mathbb{C}}$. In that case we would define $\mathcal{O}(2)$ as the pullback to \mathcal{Z} of the line bundle determined by the character $\exp \underline{H}$ of $P_{\mathbb{C}}$. This definition makes it easy to see that $\mathcal{O}(2)$ is a holomorphic line bundle; it is what we will use in the discussion of quaternionic discrete series below.

More generally, define $\mathcal{O}(m) = \mathcal{O}(2)^{\otimes \frac{m}{2}}$ for m even; this is similarly a Hermitian holomorphic line bundle over \mathcal{Z} .

¹¹By definition, the character “ $\exp i\underline{J}_3$ ” takes the value $\exp i\underline{J}_3$ on $\exp(\underline{J}_3 J_3 + \underline{S} S) \in H$; we will use this notation for characters frequently, always implicitly with respect to some basis of the corresponding Lie algebra.

The Swann space \mathcal{S} , also known as the hyperkähler cone over $K \setminus G$, is the total space of $\mathcal{O}(-2)$ over \mathcal{Z} ; locally it could be parameterized by complex coordinates on \mathcal{Z} plus one additional coordinate in the fiber of $\mathcal{O}(-2)$. It is naturally a hyperkähler manifold, with an $SU(2)$ isometry rotating the complex structures into one another [48, 47].

The circle in the unit circle bundle of $\mathcal{O}(-2)$ “cancels against” the $U(1)$ in the denominator of (2.32). So this circle bundle is the homogeneous 3-Sasakian space $U(1) \setminus SU(2, 1)$, which can be parameterized by the coset representative

$$e_{3S} = e^{i\phi J_3} e_{\mathcal{Z}} \quad (2.64)$$

\mathcal{S} is then a real cone over this homogeneous space,

$$\mathcal{S} = \mathbb{R}^+ \times U(1) \setminus SU(2, 1) \quad (2.65)$$

The right-invariant form $\theta_{3S} = de_{3S}e_{3S}^{-1}$ has J_3 component

$$\underline{J}_3 = d\phi + \frac{i}{1+z\bar{z}} \left(\bar{z}dz - zd\bar{z} + 2(\bar{z}u - z\bar{u}) + \frac{1}{2}(v - \bar{v})(1 - z\bar{z}) \right) := D\phi \quad (2.66)$$

while the other components are identical to those of $\theta_{\mathcal{Z}}$ except for a rotation under $U(1)_{J_3}$. Note that we can rewrite

$$\underline{J}_3 = d\phi + \frac{i}{1+z\bar{z}} (\bar{z}dz - zd\bar{z}) + x_a A^a \quad (2.67)$$

The metric of the 3-Sasakian space is obtained by adding \underline{J}_3^2 to (2.38), with the appropriate coefficient to enforce $SU(2)$ (left) invariance. The metric on \mathcal{S} is therefore

$$ds_{\mathcal{S}}^2 = -[dr^2 + r^2(D\phi^2 - ds_{\mathcal{Z}}^2)] \quad , \quad (2.68)$$

with indefinite signature (4,4).

2.4 Quasi-conformal representations

2.4.1 Principal series

An interesting family of “principal series” representations of G are obtained by induction from the parabolic subgroup P generated by $\{F, F_p, F_q, H, J\}$, using the character $\chi_k = e^{-kH/2}$ of P for some $k \in \mathbb{C}$. We now briefly recall the definition of induction, which we will use many times in this paper; see *e.g.* [49] for more.

The representation space of the induced representation consists of functions f on G which obey

$$f(gp) = \chi_k(p)f(g). \quad (2.69)$$

These functions can also be thought of as representing sections of a homogeneous line bundle over $P \backslash G$ defined by the character χ_k . We represent them concretely by choosing specific representatives of $P \backslash G$; a simple choice is to use elements of the opposite nilpotent radical \bar{N} generated by E_p, E_q, E , *i.e.* upper-triangular matrices.

So to compute concretely the action of some $E \in \mathfrak{g}$, we act on the upper-triangular matrix

$$\begin{pmatrix} 1 & \tilde{\zeta} + i\zeta & i\sigma - \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2) \\ & 1 & -\tilde{\zeta} + i\zeta \\ & & 1 \end{pmatrix} \quad (2.70)$$

by E and then act from the left by a suitable $X \in \mathfrak{p}$ to put the result back in upper triangular form. This gives a differential operator acting on $(\zeta, \tilde{\zeta}, \sigma)$, to which we must add $\chi_k(X)$ reflecting the twist by (2.69). The result is

$$E^{QC} = -\partial_\sigma, \quad E_p^{QC} = \partial_{\tilde{\zeta}} + \zeta \partial_\sigma, \quad E_q^{QC} = -\partial_\zeta + \tilde{\zeta} \partial_\sigma, \quad (2.71a)$$

$$H^{QC} = -\tilde{\zeta} \partial_{\tilde{\zeta}} - \zeta \partial_\zeta - 2\sigma \partial_\sigma - k, \quad J^{QC} = -\tilde{\zeta} \partial_\zeta + \zeta \partial_{\tilde{\zeta}} \quad (2.71b)$$

$$F_p^{QC} = \frac{1}{2}(3\zeta^2 - \tilde{\zeta}^2) \partial_{\tilde{\zeta}} + (\sigma - 2\tilde{\zeta} \zeta) \partial_\zeta - \frac{1}{2} \left[\zeta(\tilde{\zeta}^2 + \zeta^2) + 2\sigma \tilde{\zeta} \right] \partial_\sigma - k \tilde{\zeta} \quad (2.71c)$$

$$F_q^{QC} = -\frac{1}{2}(3\tilde{\zeta}^2 - \zeta^2) \partial_\zeta + (\sigma + 2\tilde{\zeta} \zeta) \partial_{\tilde{\zeta}} - \frac{1}{2} \left[\tilde{\zeta}(\zeta^2 + \tilde{\zeta}^2) - 2\sigma \zeta \right] \partial_\sigma + k \zeta \quad (2.71d)$$

$$F^{QC} = \frac{1}{2} \left[\zeta(\tilde{\zeta}^2 + \zeta^2) + 2\sigma \tilde{\zeta} \right] \partial_{\tilde{\zeta}} - \frac{1}{2} \left[\tilde{\zeta}(\zeta^2 + \tilde{\zeta}^2) - 2\sigma \zeta \right] \partial_\zeta - \frac{1}{4} \left[(\tilde{\zeta}^2 + \zeta^2)^2 - 4\sigma^2 \right] \partial_\sigma + k\sigma \quad (2.71e)$$

The quadratic and cubic Casimirs are constants,

$$C_2 = -k(4 - k)/4, \quad C_3 = 0 \quad (2.72)$$

corresponding to $p = q = (k - 4)/2$.

If we choose $k = 2 + is$ for $s \in \mathbb{R}$, then this representation is infinitesimally unitary with respect to the L^2 inner product with measure $d\zeta \, d\tilde{\zeta} \, d\sigma$.

2.4.2 Quaternionic discrete series

Above we constructed the principal series representations using the action of G on appropriate sections of line bundles over $P \backslash G$. There is a complex-analytic analogue of this construction, described in [26], which uses instead the action of G on \mathcal{Z} .

Formally the construction is easy to understand. We gave the action of G on holomorphic functions on \mathcal{Z} above in (2.56a). It may be slightly generalized to give formally a representation on holomorphic sections of the homogeneous line bundle $\mathcal{O}(-k)$ over

\mathcal{Z} ,

$$E^{QC} = -\partial_\alpha, \quad E_p^{QC} = \partial_{\tilde{\xi}} + \xi \partial_\alpha, \quad E_q^{QC} = -\partial_\xi + \tilde{\xi} \partial_\alpha, \quad (2.73a)$$

$$H^{QC} = -\tilde{\xi} \partial_{\tilde{\xi}} - \xi \partial_\xi - 2\alpha \partial_\alpha - k, \quad J^{QC} = -\tilde{\xi} \partial_\xi + \xi \partial_{\tilde{\xi}} \quad (2.73b)$$

$$F_p^{QC} = \frac{1}{2}(3\xi^2 - \tilde{\xi}^2) \partial_{\tilde{\xi}} + (\alpha - 2\tilde{\xi}\xi) \partial_\xi - \frac{1}{2} \left[\xi(\tilde{\xi}^2 + \xi^2) + 2\alpha\tilde{\xi} \right] \partial_\alpha - k\tilde{\xi} \quad (2.73c)$$

$$F_q^{QC} = -\frac{1}{2}(3\tilde{\xi}^2 - \xi^2) \partial_\xi + (\alpha + 2\tilde{\xi}\xi) \partial_{\tilde{\xi}} - \frac{1}{2} \left[\tilde{\xi}(\xi^2 + \tilde{\xi}^2) - 2\alpha\xi \right] \partial_\alpha + k\xi \quad (2.73d)$$

$$F^{QC} = \frac{1}{2} \left[\xi(\tilde{\xi}^2 + \xi^2) + 2\alpha\tilde{\xi} \right] \partial_{\tilde{\xi}} - \frac{1}{2} \left[\tilde{\xi}(\xi^2 + \tilde{\xi}^2) - 2\alpha\xi \right] \partial_\xi \quad (2.73e)$$

$$- \frac{1}{4} \left[(\tilde{\xi}^2 + \xi^2)^2 - 4\alpha^2 \right] \partial_\alpha + k\alpha$$

The differential operators (2.73) are of course simply related to the ones we wrote above in (2.71), by replacing the real coordinates $(\zeta, \tilde{\zeta}, \sigma)$ on $P \setminus G$ by the complex $(\xi, \tilde{\xi}, \alpha)$ on $P_{\mathbb{C}} \setminus G_{\mathbb{C}}$.

This representation is formally unitary with respect to the inner product

$$\langle f_1 | f_2 \rangle = \int_{\mathcal{Z}} d\xi d\tilde{\xi} d\alpha d\bar{\xi} d\bar{\alpha} e^{(k-4)K_{\mathcal{Z}}} f_1^*(\bar{\xi}, \bar{\xi}, \bar{\alpha}) f_2(\xi, \tilde{\xi}, \alpha) \quad (2.74)$$

Indeed, the invariant volume form on \mathcal{Z} is $e^{-4K_{\mathcal{Z}}} |d\xi d\tilde{\xi} d\alpha|^2$ (more generally -4 in the exponent would be replaced by $-2d - 2$, where d is the dimension of $K \setminus G$) while the factor $e^{kK_{\mathcal{Z}}}$ comes from the Hermitian norm in $\mathcal{O}(-k)$ [36].

However, this representation would appear to be trivial, since (for $k > 0$) there are no global sections of this line bundle, i.e. the zero-th cohomology $H^0(\mathcal{Z}, \mathcal{O}(-k))$ is empty. In [26] the desired representation space is identified instead as $H^1(\mathcal{Z}, \mathcal{O}(-k))$. Here we work directly with holomorphic sections possessing some singularities, which should be understood as Čech representatives for classes in $H^1(\mathcal{Z}, \mathcal{O}(-k))$ with respect to coverings by two open sets, in the spirit of the early literature on twistor theory. It is not obvious that one obtains all classes in $H^1(\mathcal{Z}, \mathcal{O}(-k))$ in this way, but in most of our considerations we restrict ourselves to these, and indeed we simply write $f \in H^1(\mathcal{Z}, \mathcal{O}(-k))$ where f is a section with singularities.

The formal inner product (2.74) then has to be carefully interpreted, since f_1 and f_2 are Čech representatives and hence only well defined up to certain shifts by holomorphic functions; in order to get a well defined inner product, one must interpret (2.74) in a way that involves only contour integrals. Here we give only a formal heuristic account, which will be adequate for our purposes in Section 2.4.5. We begin by analytically continuing $K_{\mathcal{Z}}$ to a function on $\mathcal{Z} \times \bar{\mathcal{Z}}$, obtained by considering the holomorphic and antiholomorphic dependence independently. Then for any $f_1 \in H^1(\mathcal{Z}, \mathcal{O}(-k))$ we define

$\hat{f}_1 \in H^1(\bar{\mathcal{Z}}, \bar{\mathcal{O}}(k-4))$ by

$$\hat{f}_1 = \int d\xi d\tilde{\xi} d\alpha e^{(k-4)K_{\mathcal{Z}}} f_1, \quad (2.75)$$

where the integral runs over some contour in $\mathcal{Z} \times \bar{\mathcal{Z}}$. This construction is an analogue of the “twistor transform” discussed in *e.g.* [50] for the case $\mathcal{Z} = \mathbb{CP}^3$. In [50] it is argued that this transform is involutive, $\hat{\hat{f}} = f$; in Section 2.4.5 we will assume that the same is true in the present case. (This is an analogue of the fact that on a Hermitian symmetric space the Kähler potential behaves as a reproducing kernel for holomorphic functions.)

Complex conjugation gives $\bar{\hat{f}}_1 \in H^1(\mathcal{Z}, \mathcal{O}(k-4))$, which can be paired with $f_2 \in H^1(\mathcal{Z}, \mathcal{O}(-k))$ and contour-integrated, so that

$$\langle f_1 | f_2 \rangle = \int d\xi d\tilde{\xi} d\alpha \bar{\hat{f}}_1 f_2. \quad (2.76)$$

This is our interpretation of (2.74). It is still formal, since we did not specify the contours of integration. In the example we consider below there will be a natural choice. In general, however, it is difficult to make sense of this formal prescription, much less to check that it is positive definite; one instead checks the existence of a positive definite norm by a purely algebraic computation on a special basis of “elementary states” (in our context, these are the K -finite vectors).

We do not perform such an analysis here, but rely on the results of [26]. There one finds that for $k \geq 2$ the spaces $H^1(\mathcal{Z}, \mathcal{O}(-k))$ are irreducible and unitarizable representations of G . For $k \geq 3$ they belong to the discrete series and are called quaternionic discrete series representations. The representation at $k = 2$ is a limit of the quaternionic discrete series.

2.4.3 Quaternionic discrete series as subquotients of principal series

These quaternionic discrete series representations are expected to be obtained as subquotients of the principal series which we discussed above. To understand why this happens, we recall the simpler case of the unitary representations of $SL(2, \mathbb{R})$. There one has a continuous principal series realized in a space of sections of a line bundle \mathcal{L}_k ($k \in \mathbb{C}$) over the equator of $\mathbb{CP}^1 = B_+ \backslash SL(2, \mathbb{C})$. If $k \in \mathbb{Z}$, then \mathcal{L}_k extends holomorphically over the whole of \mathbb{CP}^1 . One then gets the holomorphic and antiholomorphic discrete series by looking at sections which admit analytic continuation from the equator over respectively the northern or southern hemisphere, then dividing out by the space of sections which can be continued over the whole of \mathbb{CP}^1 .

In the quaternionic case the situation is similar. Firstly, the space $P \backslash G$ occurs as part of the boundary of \mathcal{Z} in an appropriate sense; in terms of our coordinates

$(\xi, \tilde{\xi}, \alpha)$ on \mathcal{Z} , this boundary is the locus where all coordinates become real (then they are identified with our coordinates on $P \backslash G$ by $\xi \rightarrow \zeta$, $\tilde{\xi} \rightarrow \tilde{\zeta}$, $\alpha \rightarrow \sigma$.) So for discrete values of k , one might expect to obtain a submodule of the principal series representation by considering those sections which are boundary values of holomorphic objects on \mathcal{Z} , and then obtain a unitary representation as some quotient thereof.¹² We will see this expectation realized in the algebraic discussion of the next subsection.

2.4.4 K -type decomposition

We now discuss the decomposition of the principal series representation under the maximal compact subgroup $K = SU(2) \times U(1)$, which we recall is generated by J_{\pm} , J_3 and S .

We begin by constructing the spherical vector, invariant under K . For this purpose consider the action of K from the right on our coset representative for $P \backslash G$,

$$n = \begin{pmatrix} 1 & \tilde{\zeta} + i\zeta & i\sigma - \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2) \\ & 1 & -\tilde{\zeta} + i\zeta \\ & & 1 \end{pmatrix}. \quad (2.77)$$

K acts on the three rows v_i , preserving their Hermitian norms $\|v_i\|^2$. On the other hand, the action of P from the left mixes the rows. Since P is lower-triangular, though, its action on the top row is simple: it just acts by the character $e^{\underline{H} + i\underline{J}/3}$. Now consider the function

$$f_K = \|v_1\|^{-k/2} = \left(1 + \tilde{\zeta}^2 + \zeta^2 + \sigma^2 + \frac{1}{4}(\tilde{\zeta}^2 + \zeta^2)^2 \right)^{-k/2} \quad (2.78)$$

as an element in the principal series representation. By definition, to compute the action of $k \in K$ on f_K , we first transform n by k acting from the right, then act by a compensating element $p \in P$ from the left to restore the form (2.77). This modifies (2.78) by a factor $e^{-k\underline{H}(p)/2}$. However, we also have to include the explicit factor $e^{k\underline{H}(p)/2}$ from the definition of the principal series. So altogether we find that (2.78) is K -invariant.

¹²The notion of “boundary value” in this case is somewhat subtler than it was in the case of $SL(2, \mathbb{R})$, because on \mathcal{Z} we deal not with functions but with cohomology classes. The reason is that the structure of \mathcal{Z} near the boundary is more complicated than that of the upper half-plane; it is not contained in any convex tube domain, essentially because of the circle parameterized by the phase of z , which winds around the boundary. The correct notion of boundary value in this case should involve integration over this circle, as described in *e.g.* [51].

More generally, the highest weight vectors of $SU(2)_J$ (vectors annihilated by J_+) are given by

$$f_{j,s} = \left[\frac{\tilde{\zeta} + i\zeta}{1 + i\sigma + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)} \right]^{j - \frac{1}{3}s} \left[\frac{1 + i\sigma - \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)}{1 - i\sigma + \frac{1}{2}(\zeta^2 + \tilde{\zeta}^2)} \right]^{j + \frac{1}{3}s} \times \left[1 + \tilde{\zeta}^2 + \zeta^2 + \sigma^2 + \frac{1}{4}(\tilde{\zeta}^2 + \zeta^2)^2 \right]^{-k/2} \quad (2.79)$$

with eigenvalues ji and si for J_3 and S , respectively.

In section 2.6.1 below, we shall see that these states have a simple expression in terms of the moment maps of the action of G on the symplectization of $P \backslash G$. For now, we observe that the highest weight states $f_{j,s}$ are mapped to each other by the action of the raising operators $J_{\frac{1}{2}, \pm \frac{3}{2}}$:

$$\begin{aligned} J_{\frac{1}{2}, \frac{3}{2}} \cdot f_{j,s} &= \frac{1}{3}(3k + 6j + 2s) f_{j+\frac{1}{2}, s+\frac{3}{2}} \\ J_{\frac{1}{2}, -\frac{3}{2}} \cdot f_{j,s} &= \frac{2\sqrt{2}}{3}(3k + 6j - 2s) f_{j+\frac{1}{2}, s-\frac{3}{2}} \end{aligned} \quad (2.80)$$

Applying the lowering operators $J_{-\frac{1}{2}, \pm \frac{3}{2}}$ gives a linear combination of the highest weight state $f_{j-\frac{1}{2}, s \pm \frac{3}{2}}$ and a descendant of $f_{j+\frac{1}{2}, s \pm \frac{3}{2}}$:

$$\begin{aligned} J_{-\frac{1}{2}, \frac{3}{2}} \cdot f_{j,s} &= \frac{(3j - s)(6 + 6j - 3k - 2s)}{18\sqrt{2}(2j + 1)} f_{j-\frac{1}{2}, s+\frac{3}{2}} + \frac{3k + 6j + 2s}{3(2j + 1)} J_{-1} \cdot f_{j+\frac{1}{2}, s+\frac{3}{2}} \\ J_{-\frac{1}{2}, -\frac{3}{2}} \cdot f_{j,s} &= \frac{(3j + s)(6 + 6j - 3k + 2s)}{9(2j + 1)} f_{j-\frac{1}{2}, s-\frac{3}{2}} + 2\sqrt{2} \frac{3k + 6j - 2s}{3(2j + 1)} J_{-1} \cdot f_{j+\frac{1}{2}, s-\frac{3}{2}} \end{aligned} \quad (2.81)$$

Using these equations, we may now study the structure of the module generated by $f_{j,s}$ and its descendants; it is pictured in Figure 2.

For generic k this module is irreducible and not manifestly unitarizable. When k is an integer, the situation is more interesting. For even integers $k \geq 2$, there is an irreducible submodule generated by $f_{j=(k-2)/2, s=0}$. Its K -type decomposition coincides with that of the representation labeled by $p = q = (k - 4)/2$ in the parameterization of [6], namely,

$$\bigoplus_{m=k-2}^{\infty} \bigoplus_{s=-3m/2}^{3m/2} [j = m/2]_s \quad (2.82)$$

We identify it as the quaternionic discrete series with index k (or limit discrete series for $k = 2$) [26]; in particular, it is unitarizable. It has no spherical vector unless $k = 2$.

For (possibly negative) even integers $k \leq 2$, we can similarly obtain the representation with $p = q = -k/2$, this time as a quotient instead of a submodule; namely, we divide out the submodule consisting of all states with $3j - |s| < 3(k - 2)/2$. The resulting representations are equivalent to the ones just discussed.

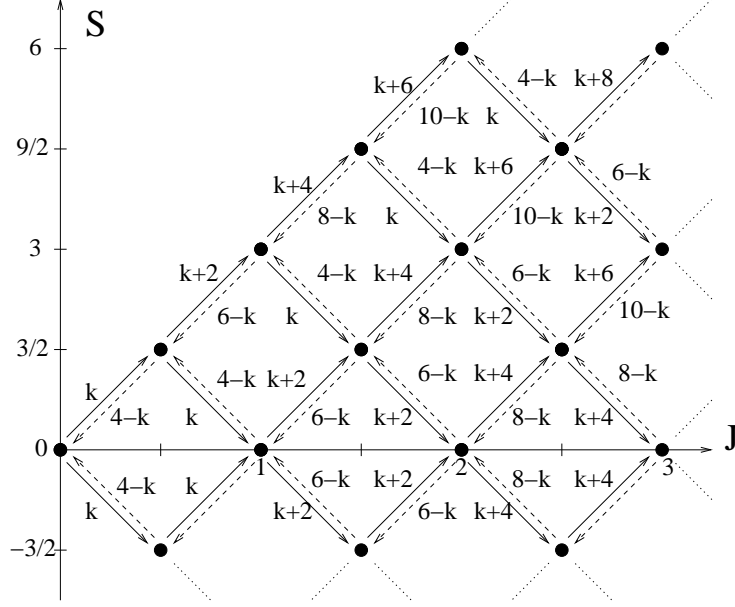


Figure 2: Structure of the module generated by the highest weights $f_{j,s}$. The solid (resp. dotted) arrows denote the action of the raising (resp. lowering) operators, with coefficient proportional to the indicated function of k .

In [26] one also finds quaternionic discrete series representations for odd k ; one might wonder why we did not encounter those above. The answer is that strictly speaking they are not representations of $G = SU(2, 1)$ but rather of its double cover. Correspondingly, they do not appear as subrepresentations of the principal series we considered here, but of a closely related “non-spherical principal series” representation of the double cover.

2.4.5 Matrix elements and Penrose transform

Suppose ρ is a spherical unitary representation of G , with spherical vector f_K . Then ρ can be realized in the space of functions on $K \backslash G$, by mapping any state f to the matrix element

$$\varphi(e_{QK}) = \langle f | \rho(e_{QK}^{-1}) f_K \rangle \quad (2.83)$$

φ is a function on $K \backslash G$ because the left action of $k \in K$ on e_{QK} becomes a right action on e_{QK}^{-1} , hence a left action on f_K , which is trivial because f_K is spherical. Moreover,

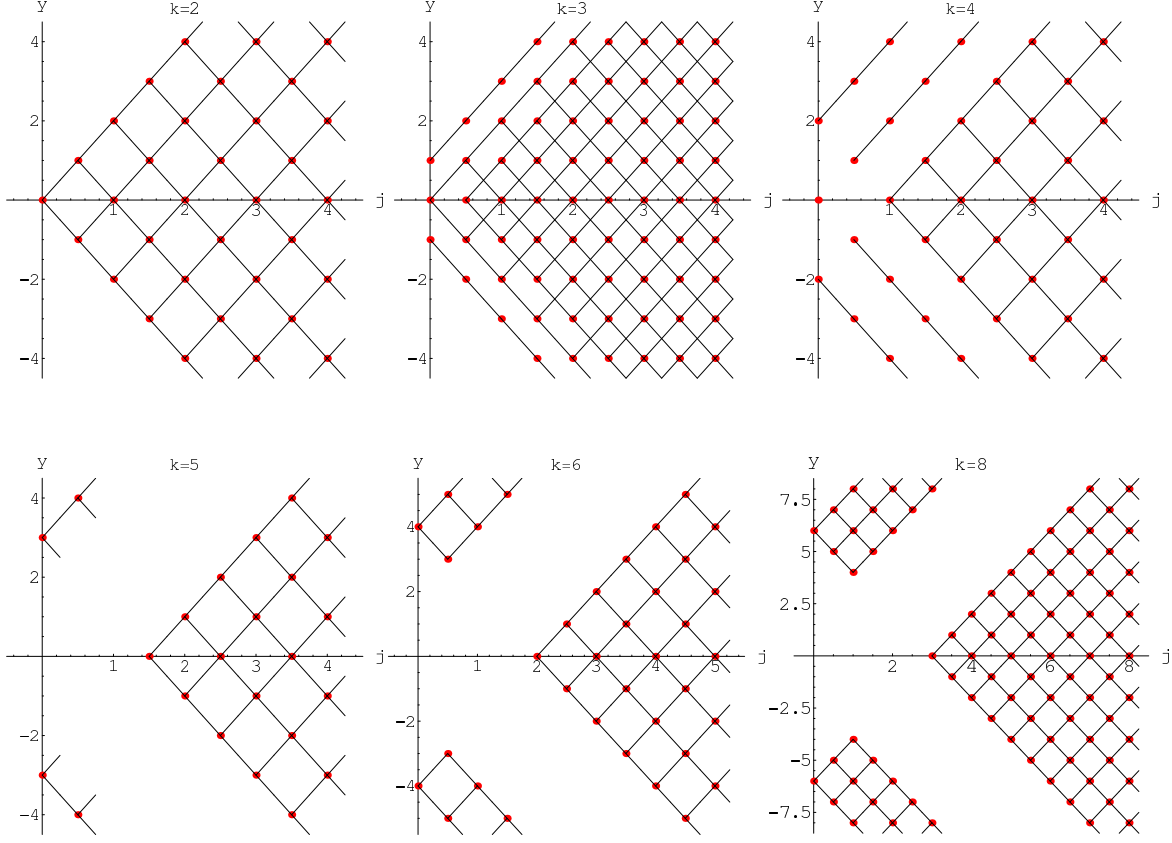


Figure 3: K -type decomposition of the discrete series representations of $SU(2,1)$ in the $(j, y = 2/3s)$ plane, for low values of $k = (p - 4)/2 = (q - 4)/2$. The quaternionic discrete series corresponds to the “ $p + q$ discrete branch” in the terminology of [6]. The “ p -discrete” branch for $k = 3$ corresponds to the minimal representation, see Section 2.5.2. The second “ p -discrete branch” for $k = 4$, starting at $(j, s) = (1/2, \pm 3/2)$, corresponds to the deformed minimal representation at $\nu = \pm 1$. For $k = 5$ and higher odd values of k , there are no “ p -discrete” or “ q -discrete” branches, as the lowering operators $J_{\frac{1}{2}, \pm \frac{3}{2}}$ map into forbidden regions, as illustrated by the open links.

the φ so obtained obeys differential equations determined by ρ ; in particular, it is an eigenfunction of the Laplacian on $K \backslash G$ with eigenvalue $2C_2(\rho)$.

Even if ρ is not spherical one may still apply this construction replacing f_K by any K -finite vector, and thus embed ρ into a space of sections of a homogeneous vector bundle over $K \backslash G$, induced from the representation of K in which f_K transforms. The sections so obtained have particularly good properties if f_K is in the lowest K -type (e.g. for holomorphic discrete series representations they turn out to be holomorphic

sections).

We now apply this construction to the quaternionic discrete series representations, beginning with the special case $k = 2$. Using the Iwasawa decomposition (2.22) and the Baker-Campbell-Hausdorff formula $e^A e^B = e^B e^A e^{[A,B]}$ when $[A, B]$ is central,

$$e_{QK}^{-1} = e^{-(\sigma - \zeta \tilde{\zeta})E} \cdot e^{\zeta E_q} \cdot e^{-\tilde{\zeta} E_p} \cdot e^{-UH} \quad (2.84)$$

and

$$\rho(e_{QK}^{-1}) = e^{-(\sigma - \tilde{\xi} \zeta + \xi \tilde{\zeta})\partial_\alpha} \cdot e^{-\zeta \partial_\xi} \cdot e^{-\tilde{\zeta} \partial_{\tilde{\xi}}} \cdot e^{U(\partial_U + \xi \partial_\xi + \tilde{\xi} \partial_{\tilde{\xi}} + 2\alpha \partial_\alpha + k)}. \quad (2.85)$$

Applying this $\rho(e_{QK}^{-1})$ to the spherical vector $f_K = f_{0,0}$ yields

$$\begin{aligned} \Psi(U, \zeta, \tilde{\zeta}, \sigma; \xi, \tilde{\xi}, \alpha) = & \left\{ e^{2U} + (\tilde{\xi} - \tilde{\zeta})^2 + (\xi - \zeta)^2 \right. \\ & \left. + e^{-2U} \left[(\sigma + \alpha + \tilde{\xi} \zeta - \xi \tilde{\zeta})^2 + \frac{1}{4} \left((\tilde{\xi} - \tilde{\zeta})^2 + (\xi - \zeta)^2 \right)^2 \right] \right\}^{-1} \end{aligned} \quad (2.86)$$

Since $C_2(\rho) = -1$, taking the inner product (2.83) between f and (2.86) yields an eigenfunction of the conformal Laplacian on $K \backslash G$,

$$[\Delta_{QK} + 2] \varphi = 0 \quad (2.87)$$

Thus the transformed spherical vector (2.86) is the kernel for an integral operator which intertwines between the quaternionic discrete series and a subspace of the space of functions on $K \backslash G$. In physics parlance, it is related to the boundary to bulk propagator for (2.87), upon taking the coordinates $\xi, \tilde{\xi}, \alpha$ to be real — in that case they lie on a part of the boundary of \mathcal{Z} , which may be identified with a boundary of $K \backslash G$. (See [52] for an attempt to set up a bulk-boundary correspondence on this space.)

Since $K \backslash G$ is a quaternionic-Kähler manifold there is an *a priori* different way to construct a function on $K \backslash G$ from $f(\xi, \tilde{\xi}, \alpha) \in H^1(\mathcal{Z}, \mathcal{O}(-2))$, which is to apply the quaternionic Penrose transform [53, 54]. Some details of this correspondence for quaternionic-Kähler spaces obtained by the c -map construction were worked out in [36], where a simple contour integral formula for the Penrose transform was established:

$$\text{Penrose}[f] = \varphi(U, \zeta, \tilde{\zeta}, \sigma) = 2 e^{2U} \oint \frac{dz}{z} f \left[\xi(z), \tilde{\xi}(z), \alpha(z) \right] \quad (2.88)$$

The resulting φ is annihilated by the conformal Laplacian $\Delta_{QK} + 2$. This agrees with (2.87) in the case of $K \backslash G$, and in this context it can be checked directly by acting with Δ_{QK} on the integrand of (2.88) and using (2.71):

$$\left[\Delta_{QK} + 2 - \partial_z \left(\frac{z^2}{2} \partial_z + \frac{1}{2} z \partial_z + 2i e^{2U} z \partial_\alpha \right) \right] z^{-1} f \left[\xi(z), \tilde{\xi}(z), \alpha(z) \right] = 0 \quad (2.89)$$

Integration over z eliminates the total derivative, leading to (2.87).

We now argue that these two constructions are in fact identical, i.e.

$$\langle f | \rho(e_{QK}^{-1}) f_K \rangle = \text{Penrose}[f]. \quad (2.90)$$

According to our discussion of the inner product above, this is equivalent to

$$\int d\xi_0 d\tilde{\xi}_0 d\alpha_0 f \Psi = \text{Penrose}[\hat{f}], \quad (2.91)$$

where $\Psi = \rho(e_{QK}^{-1}) f_K$; then using (2.75) and the explicit form of $K_{\mathcal{Z}}$, (2.91) is equivalent to requiring that Ψ arises as the Penrose transform of

$$\Phi(\xi, \tilde{\xi}, \alpha) = \left[(\alpha - \alpha_0 + \xi_0 \xi - \tilde{\xi}_0 \xi_0)^2 + \frac{1}{4} [(\xi - \xi_0)^2 + (\tilde{\xi} - \tilde{\xi}_0)^2]^2 \right]^{-1} \quad (2.92)$$

But this we can evaluate directly: the contour integral (2.88) defining $\text{Penrose}[\Phi]$ has poles at $z = z_{\pm}$, where

$$z_+ = \frac{2i\sqrt{2}e^U \left[(\zeta - \xi_0) - i(\tilde{\zeta} - \tilde{\xi}_0) \right]}{2i(\sigma - \alpha_0 + \tilde{\xi}_0 \zeta - \tilde{\zeta} \xi_0) + (\zeta - \xi_0)^2 + (\tilde{\zeta} - \tilde{\xi}_0)^2 - 2e^{2U}} \quad (2.93)$$

and $z_- = -1/\bar{z}_+$. The residue at $z = z_{\pm}$ yields

$$\begin{aligned} \Psi = & \left\{ e^{2U} + (\zeta - \xi_0)^2 + (\tilde{\zeta} - \tilde{\xi}_0)^2 \right. \\ & \left. + e^{-2U} \left[(\sigma - \alpha_0 + \tilde{\xi}_0 \zeta - \tilde{\zeta} \xi_0)^2 + \frac{1}{4} [(\zeta - \xi_0)^2 + (\tilde{\zeta} - \tilde{\xi}_0)^2]^2 \right] \right\}^{-1} \end{aligned} \quad (2.94)$$

which indeed agrees with the formula (2.86) for $\rho(e_{QK}^{-1}) f_K$ as desired.

Similar considerations apply for other even values of k . In that case the Penrose transform gives a section of $Sym^{k-2}(H)$; for c -map spaces, in a natural trivialization of H , the formula is given in [36] as

$$\varphi_m(U, \zeta, \tilde{\zeta}, \sigma) = 2 e^{kU} \oint \frac{dz}{z} z^{\frac{1}{2}m} f \left[\xi(z), \tilde{\xi}(z), \alpha(z) \right] \quad (2.95)$$

where $m = -k + 2, \dots, k - 2$ labels the $2k - 3$ components of φ . This turns out to agree with the matrix element construction, where we now use the $(2k - 3)$ -dimensional lowest K -type of the quaternionic discrete series. Establishing a similar correspondence for k odd would require a better understanding of the branch cuts appearing in the contour integral (2.88).

2.4.6 Causal structure and quartic light-cone

A general fact about twistor spaces of four-dimensional conformally self-dual manifolds is that two points x, x' are lightlike separated if and only if their corresponding twistor lines $L_x, L_{x'}$ intersect in \mathcal{Z} . (Since $K \setminus G$ has Euclidean signature, we must of course allow x and x' to belong to its complexification if this condition is to be satisfied.)

Using the twistor map (2.58), the condition for $(U, \zeta, \tilde{\zeta}, \sigma)$ and $(U', \zeta', \tilde{\zeta}', \sigma')$ to be light-like separated is therefore that there exist z and z' such that

$$\begin{aligned} \zeta - \frac{i}{\sqrt{2}}e^U (z + z^{-1}) &= \zeta' - \frac{i}{\sqrt{2}}e^{U'} (z' + z'^{-1}) \\ \tilde{\zeta} + \frac{1}{\sqrt{2}}e^U (z - z^{-1}) &= \tilde{\zeta}' + \frac{1}{\sqrt{2}}e^{U'} (z' - z'^{-1}) \\ \sigma + \frac{1}{\sqrt{2}}e^U \left[z(\zeta + i\tilde{\zeta}) + z^{-1}(-\zeta + i\tilde{\zeta}) \right] &= \sigma' + \frac{1}{\sqrt{2}}e^{U'} \left[z'(\zeta' + i\tilde{\zeta}') + z'^{-1}(-\zeta' + i\tilde{\zeta}') \right] \end{aligned} \quad (2.96)$$

Eliminating z and z' from the first two equations, we find

$$z = \frac{i e^{-U}}{2\sqrt{2}(\Delta\zeta + i\Delta\tilde{\zeta})} \left(2(e^{2U'} - e^{2U}) + \Delta\zeta^2 - \Delta\tilde{\zeta}^2 + 2\sqrt{\Delta_4} \right) \quad (2.97a)$$

$$z' = \frac{i e^{-U'}}{2\sqrt{2}(\Delta\zeta + i\Delta\tilde{\zeta})} \left(2(e^{2U} - e^{2U'}) + \Delta\zeta^2 + \Delta\tilde{\zeta}^2 + 2\sqrt{\Delta_4} \right) \quad (2.97b)$$

where we denoted

$$\Delta\zeta = \zeta' - \zeta, \quad \Delta\tilde{\zeta} = \tilde{\zeta}' - \tilde{\zeta}, \quad \Delta\sigma = \sigma' - \sigma + \zeta'\tilde{\zeta} - \zeta\tilde{\zeta}' \quad (2.98)$$

and

$$\Delta_4 = \frac{1}{4}(\Delta\zeta^2 + \Delta\tilde{\zeta}^2)^2 + (e^{2U} + e^{2U'}) (\Delta\zeta^2 + \Delta\tilde{\zeta}^2) + (e^{2U} - e^{2U'})^2 \quad (2.99)$$

Reinserting in the third equation in (2.96), we find

$$\Delta \equiv (\Delta\sigma)^2 + \frac{1}{4} \left(\Delta\zeta^2 + \Delta\tilde{\zeta}^2 \right)^2 + (e^{2U} + e^{2U'}) \left(\Delta\zeta^2 + \Delta\tilde{\zeta}^2 \right) + (e^{2U} - e^{2U'})^2 = 0 \quad (2.100)$$

For $U, U' \rightarrow -\infty$, we recognize the familiar quartic norm N_4 from (2.57), now evaluated at real values of its arguments. If only U' is sent to $-\infty$, we recover the “transformed spherical vector” (2.86). Expanding to quadratic order in the variations (2.98), we find

$$\Delta \sim 2e^{4U} \left[2dU^2 + e^{-2U} \left(d\tilde{\zeta}^2 + d\zeta^2 \right) + \frac{1}{2}e^{-4U} \left(d\sigma + \tilde{\zeta}d\zeta - \zeta d\tilde{\zeta} \right)^2 \right] \quad (2.101)$$

which indeed vanishes for infinitesimal light-like displacements under the metric (2.28).

Since we are discussing issues of causal structure and group theory, we make a side comment here. The groups $SU(2, n)$ are the only quaternionic groups that admit positive energy unitary representations. The $U(1)$ generator in the maximal compact subgroup $U(1) \times SU(n)$ is the generator whose spectrum is bounded from below for such representations and hence can be taken as the Hamiltonian for a causal time evolution. The other quaternionic groups do not have any generators whose spectrum is bounded from below for any unitary representation. Hence for these groups the Hamiltonian with a bounded spectrum that describes causal evolution can not be one of the generators.

2.5 Minimal representation

For any real Lie group G there is a notion of “minimal unitary representation” introduced in [55] and much studied thereafter (see *e.g.* [28] for a recent review.) For many G the minimal representation can be characterized as the unitary representation of smallest Gelfand-Kirillov dimension. For $G = SU(2, 1)$ the situation is somewhat degenerate and there are many representations sharing this minimal dimension, including the holomorphic and antiholomorphic discrete series¹³. We focus here on the representation constructed explicitly in [25] by truncation of the minimal unitary realization of $E_{8(8)}$ and whose structure parallels that of the minimal representation of higher rank groups. With a slight change of notation and normalization relative to this reference, the generators can be written as

$$E_p = i x u , \quad F_p = i \partial_u \partial_x + \frac{i}{2x} (u^3 - \partial_u u \partial_u) \quad (2.102a)$$

$$E_q = x \partial_u , \quad F_q = u \partial_x + \frac{i}{2x} (i \partial_u^3 - i u \partial_u u) \quad (2.102b)$$

$$E = \frac{i}{2} x^2 , \quad F = \frac{i}{2} \partial_x^2 + \frac{i}{8x^2} [(u^2 - \partial_u^2)^2 - 1] \quad (2.102c)$$

$$H = x \partial_x + \frac{1}{2} , \quad J = \frac{i}{2} (\partial_u^2 - u^2) \quad (2.102d)$$

acting on functions of two variables (u, x) . Equivalently, by defining $y = x^2$ and

¹³In fact, this is true for the entire quaternionic family $SU(n, 2)$.

$x_0 = xu$, we reach a presentation analogous to the one used in [39] for split groups,¹⁴

$$E_p = ix_0, \quad F_p = \frac{i}{2}x_0\partial_0^2 + 2iy\partial_0\partial_y + \frac{ix_0^3}{2y^2} + \frac{i}{2}\partial_0 \quad (2.103a)$$

$$E_q = y\partial_0, \quad F_q = -\frac{1}{2}y\partial_0^3 + 2x_0\partial_y + \frac{3x_0^2}{2y}\partial_0 + \frac{x_0}{2y} \quad (2.103b)$$

$$E = \frac{i}{2}y, \quad H = x_0\partial_0 + 2y\partial_y + \frac{1}{2}, \quad J = \frac{i}{2}(y\partial_0^2 - \frac{x_0^2}{y}) \quad (2.103c)$$

$$F = 2iy\partial_y^2 - i\frac{x_0^4}{8y^3} + \frac{ix_0}{2y}\partial_0 + \frac{3ix_0^2}{4y}\partial_0^2 - \frac{i}{8}y\partial_0^4 + i\partial_y + 2ix_0\partial_y\partial_0 + \frac{3i}{8y} \quad (2.103d)$$

Any minimal representation is annihilated by the Joseph ideal in the universal enveloping algebra of \mathfrak{g} . This means that the generators of the minimal representations satisfy certain quadratic identities, *e.g.*

$$H^2 + 2(EF + FE) + J^2 + 1 = 0 \quad (2.104a)$$

$$E_p^2 + E_q^2 + 4JE = 0 \quad (2.104b)$$

$$C_2(J) + \frac{1}{9}S^2 + \frac{1}{4} = 0. \quad (2.104c)$$

These hold in addition to the Casimir identities

$$C_2 = -\frac{3}{4}, \quad C_3 = 0 \quad (2.105)$$

corresponding to the parameters $(p, q) = (-\frac{1}{2}, -\frac{1}{2})$ in the classification of [6].

The generators (2.102) or (2.103) are antihermitean with respect to the inner product

$$\langle f_1 | f_2 \rangle = \int y^{-1} dy dx_0 f_1^*(y, x_0) f_2(y, x_0) = \int dx du f_1^*(u, x) f_2(u, x), \quad (2.106)$$

so this representation is unitary. For later reference, we note that the (non-normalizable) states

$$y^{\frac{1}{2}} \exp\left(\pm \frac{x_0^2}{2y}\right) = x \exp\left(\pm \frac{1}{2}u^2\right) \quad (2.107)$$

are invariant under the nilpotent radical N generated by F, F_p, F_q , and carry charges $(3/2, \pm i/2)$ under the Cartan generators (H, J) .

¹⁴The split real form of $SU(2, 1)$ is $SL(3, \mathbb{R})$.

2.5.1 Induction from the maximal parabolic and deformation

The minimal representation, in fact a one-parameter deformation thereof, can be obtained by induction from the maximal parabolic subgroup $Q \subset G_{\mathbb{C}}$ generated by $\{F, F_p, F_q, H, J, E_p + iE_q\}$. For this purpose, decompose any element of $G_{\mathbb{C}}$ into a product

$$g = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \cdot \begin{pmatrix} 1 & z & ia \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad (2.108)$$

Then induction from the character $\exp[\tau(\underline{H} + i\underline{J}/3)]$ of Q gives the action of G on sections $f(z, a)$ over $Q \backslash G_{\mathbb{C}}$ by first order differential operators,

$$E_p = \partial_z + iz\partial_a, \quad F_p = -(ia + z^2)\partial_z - az\partial_a + \tau z \quad (2.109a)$$

$$E_q = -i\partial_z - z\partial_a, \quad F_q = -(a + iz^2)\partial_z - iaz\partial_a + i\tau z \quad (2.109b)$$

$$E = \partial_a, \quad F = -az\partial_z - a^2\partial_a + \tau a \quad (2.109c)$$

$$H = -z\partial_z - 2a\partial_a + \tau, \quad J = -iz\partial_z + i\frac{\tau}{3} \quad (2.109d)$$

Set $\nu = -(2\tau + 3)/3$. Passing from $f(z, a)$ to $f(u, x)$ by the intertwining operator

$$f(z, a) = \int dudx x^{1+4\nu} e^{-\frac{1}{2}u^2 + iuxz + \frac{1}{4}x^2z^2 + \frac{i}{2}ax^2} f(u, x) \quad (2.110)$$

the action of G on $f(u, x)$ is given by a one-parameter deformation¹⁵ of the minimal representation (2.102),

$$E_p^{(\nu)} = E_p, \quad E_q^{(\nu)} = E_q, \quad E_k^{(\nu)} = E_k \quad (2.111a)$$

$$H^{(\nu)} = H + \frac{5}{2}\nu, \quad J^{(\nu)} = J - \frac{i}{2}\nu \quad (2.111b)$$

$$F_p^{(\nu)} = F_p + \nu \frac{i}{2x}(3u + 5\partial_u), \quad F_q^{(\nu)} = F_q + \nu \frac{1}{2x}(5u + 3\partial_u), \quad (2.111c)$$

$$F^{(\nu)} = F + \nu \frac{i}{4x^2} [3\partial_u^2 + 10x\partial_x + 3(1 - u^2)] + \frac{2i}{x^2}\nu(\nu - 1) \quad (2.111d)$$

with Casimirs

$$C_2 = \frac{3}{4}(\nu^2 - 1), \quad C_3 = i\nu(1 - \nu^2), \quad (2.112)$$

corresponding to

$$(p, q) = \left(-\frac{1}{2}(1 - 3\nu), -\frac{1}{2}(1 + 3\nu) \right) \quad (2.113)$$

¹⁵The fact that the minimal representation is not isolated is a peculiarity of the A series.

in the notation of [6]. The resulting representation is not obviously unitary for $\nu \neq 0$, as the inducing character $\exp[\tau(\underline{H} + i\underline{J}/3)]$ is in general not unitary. We shall however find evidence in the next section that it is unitarizable at $\nu = \pm 1$.

The annihilator of the ν -deformed minimal representation is deformed to

$$H^2 + 2(EF + FE) + J^2 - 2i\nu J + (1 - \nu^2) = 0 \quad (2.114a)$$

$$E_p^2 + E_q^2 + 4J E + 2i\nu E = 0 \quad (2.114b)$$

$$C_2(J) + \frac{1}{9}S^2 + 3i\nu S + \frac{1}{4}(1 - \nu^2) = 0 \quad (2.114c)$$

while the vectors invariant under the nilpotent radical N are deformed to

$$y^{\frac{1-\nu}{2}} \exp\left(-\frac{x_0^2}{2y}\right), \quad y^{\frac{1-4\nu}{2}} \exp\left(\frac{x_0^2}{2y}\right) \quad (2.115)$$

carrying charges $(\frac{3}{2}(1+\nu), -\frac{i}{2}(1+\nu))$ and $(\frac{3}{2}(1-\nu), \frac{i}{2}(1-\nu))$ under (H, J) , respectively.

2.5.2 K -type decomposition

For completeness, we now review the K -type decomposition of the minimal representation, i.e. the decomposition under the maximal compact subgroup $SU(2) \times U(1)$, as discussed in [25]¹⁶. For this purpose, we change polarization to oscillator representation for both the u and x variable, i.e. define a_u, a_u^\dagger, N by

$$\frac{1}{\sqrt{2}}(u - \partial_u) = a_u^\dagger, \quad \frac{1}{\sqrt{2}}(u + \partial_u) = a_u, \quad N_u = a_u^\dagger a_u \quad (2.116)$$

and similarly for x . The compact generator S is manifestly positive,

$$S = \frac{3}{8} (2N_x + 1 + x^{-2}N_u(N_u + 1) + 2N_u + 1) \quad (2.117)$$

so the representation is of lowest-weight type. The positive grade generators in the 3-grading by S read

$$K_+ = a_u a_x + \frac{\sqrt{2}}{x} (N_u + 1) a_u \quad (2.118a)$$

$$L_+ = \frac{1}{2} a_x^2 - \frac{1}{4x^2} N_u (N_u + 1) \quad (2.118b)$$

The only normalizable state annihilated by these two generators is $a_x^\dagger |0_{u,x}\rangle$, or, in the real polarization,

$$f_K = x \exp\left[-\frac{1}{2}(u^2 + x^2)\right] = y^{1/2} \exp\left[-\frac{1}{2}\left(y + \frac{x_0^2}{y}\right)\right] \quad (2.119)$$

¹⁶We should note that the positive and negative grade generators we define in this paper are opposite to those of [25].

It is easy to check that this generator is a singlet of $SU(2)_{J_{\pm}, J_3}$, but carries a non-zero charge $-3i/2$ under S . By acting with the raising operators K_- and L_- , we generate the complete K -type decomposition of the minimal representation,

$$\bigoplus_{m=0}^{\infty} \left[\frac{m}{2} \right]_{-\frac{3i}{2}(m+1)} \quad (2.120)$$

where the term in bracket is the spin of the $SU(2)_J$ representation, and the subscript indicates the S charge. This agrees with the K -type decomposition of the “ p -discrete” module at $p = q = -1/2$, as seen on Figure 3, for $k = 3$.

Let us now briefly discuss the ν -deformed minimal representation. It is easy to check that

$$f_0 = x^{1-\nu} \exp \left[-\frac{1}{2}(u^2 + x^2) \right] \quad (2.121)$$

is a singlet of $SU(2)_J$, with charge $-\frac{3i}{2}(1+\nu)$ under S , and annihilated by the deformed generators K_+, L_+ . Acting with deformed generators K_-, L_- produces a $SU(2)_J$ doublet with $S_3 = -\frac{3i}{2}(2+\nu)$,

$$f_{\frac{1}{2}} = -i u x^{2-\nu} \exp \left[-\frac{1}{2}(u^2 + x^2) \right], \quad f_{-\frac{1}{2}} = \frac{i}{2\sqrt{2}}(3 + 3\nu - 2y^2) \exp \left[-\frac{1}{2}(u^2 + x^2) \right] \quad (2.122)$$

For $\nu = -1$, both of these states are annihilated by K_+, L_+ , so generate a module corresponding to the semi-infinite line $s = -3/2j$ in the diagram on Figure 3 for $k = 4$. Thus, the K -type decomposition of the ν -deformed minimal representation at $\nu = -1$ has a ladder structure

$$\bigoplus_{m=1}^{\infty} \left[\frac{m}{2} \right]_{-\frac{3i}{2}m} \quad (2.123)$$

As usual, we can use the lowest K -type to embed the minimal representation into the space of sections of a vector bundle on $K \backslash G$, in this case a line bundle with $-3/2$ units of charge under S . For this purpose, as in (2.83), we let the coset representative e_{QK} act on the lowest K -type, and construct the intertwiner

$$\begin{aligned} \Psi(U, \zeta, \tilde{\zeta}, \sigma; u, x) &= e^{U(x\partial_x + \frac{1}{2})} \cdot e^{-i\tilde{\zeta}ux} \cdot e^{\zeta x\partial_u} \cdot e^{-\frac{i}{2}(\sigma - \zeta\tilde{\zeta})x^2} \cdot f_{SU(2)} \\ &= x \exp \left[\frac{3U}{2} - \frac{1}{2}u^2 - \frac{1}{2}(e^{2U} + i\sigma)x^2 - \frac{1}{2}x(2u + x\zeta)(\zeta + i\tilde{\zeta}) \right] \end{aligned} \quad (2.124)$$

The overlap

$$\Phi_f = \int dx du f^*(u, x) \Psi(U, \zeta, \tilde{\zeta}, \sigma; u, x) \quad (2.125)$$

is then an eigenmode of the Laplacian twisted by S ,

$$\left[\Delta_{QK} - \frac{1}{2} e^{2U} \partial_\sigma - \frac{15}{16} \right] \Phi_f = 0 . \quad (2.126)$$

2.5.3 As a submodule of the principal series representation

In this section, we investigate to what extent the minimal representation (or its ν -deformation) may be viewed as a submodule of the principal series representation.

To that purpose, we first observe that the Casimirs (2.112) and (2.72) agree for $k = 1, 3$ ($\nu = 0$) or $k = 0, 4$ ($\nu = \pm 1$). Second, the equation (2.114b) in the annihilator of the deformed minimal representation, when expressed in terms of the generators of the quasi-conformal action, becomes

$$C_0 \equiv (\partial_\zeta + \tilde{\zeta} \partial_\sigma)^2 + (\partial_{\tilde{\zeta}} - \zeta \partial_\sigma)^2 - 2i\nu \partial_\sigma = 0 \quad (2.127)$$

The physicist will recognize C_0 as the Hamiltonian of a charged particle on the plane $(\zeta, \tilde{\zeta})$, with a constant magnetic field proportional to $i\partial_\sigma$. The spectrum of C_0 (for the usual L^2 norm on the plane) consists of the usual infinitely degenerate Landau levels. Defining

$$\nabla = \partial_\zeta + \tilde{\zeta} \partial_\sigma + i(\partial_{\tilde{\zeta}} - \zeta \partial_\sigma) \quad (2.128a)$$

$$\bar{\nabla} = \partial_{\tilde{\zeta}} + \zeta \partial_\sigma - i(\partial_\zeta - \tilde{\zeta} \partial_\sigma) \quad (2.128b)$$

one may rewrite (2.127) as

$$C_0 = \nabla \bar{\nabla} - 2i(\nu + 1) \partial_\sigma = 0 \quad (2.129)$$

The lowest Landau level corresponds to functions annihilated by $\bar{\nabla}$. This constraint commutes with the action of G for $k = 0, \nu = -1$: this is evidently so for the positive root generators E_p, E_q, E (the former two being the generators of magnetic translations on the plane), and it suffices to check invariance under the action of the lowest root generator F ,

$$[F, \bar{\nabla}] = -\frac{i}{2} [3(\zeta^2 + \tilde{\zeta}^2) - 2i\sigma] \bar{\nabla} - k(\tilde{\zeta} + i\zeta) , \quad (2.130)$$

which indeed vanishes on the subspace annihilated by $\bar{\nabla}$ when $k = 0$. Solutions to $\bar{\nabla} = 0$ are of the form ¹⁷

$$f(\zeta, \tilde{\zeta}, \sigma) = g \left[z \equiv \tilde{\zeta} + i\zeta, a \equiv -\sigma + \frac{i}{2}(\zeta^2 + \tilde{\zeta}^2) \right] \quad (2.131)$$

¹⁷After Fourier transforming over σ , one recovers the usual form $f = g_K(\tilde{\zeta} + i\zeta) e^{-\frac{1}{2}K(\zeta^2 + \tilde{\zeta}^2) - iK\sigma}$ of the lowest Landau level wave functions.

It is straightforward to check that the quasi-conformal action reduces on this invariant subspace to the action (2.109) induced from the maximal parabolic at $\nu = -1$.

As far as the constraint $C_0 = 0$ itself is concerned, one may check that it is invariant under the action of G at the values $k = 1, \nu = 0$ appropriate for the undeformed minimal representation, as well as at $k = 0, \nu = -1$ which we discussed above. Indeed, one may rewrite

$$[F, C_0] = -2\sigma C_0 + 2(1 - k) \left[(\zeta^2 + \tilde{\zeta}^2) \partial_\sigma - J \right] + 2i\nu(\zeta \partial_\zeta + \tilde{\zeta} \partial_{\tilde{\zeta}} + k) \quad (2.132)$$

which is proportional to C_0 for $k = 1, \nu = 0$. From the point of view of the magnetic problem, this corresponds to non-normalizable states with energy below that of the lowest Landau level. In order to find the eigenmodes explicitly, it is useful to Fourier transform over $\tilde{\zeta}, \sigma$,

$$f(\zeta, \tilde{\zeta}, \sigma) = \int dp dK \exp \left(-iK\sigma - ip\tilde{\zeta} \right) g(\zeta, p, K) \quad (2.133)$$

and redefine

$$g(\zeta, p, K) = \exp \left[-\frac{(P - 2K\zeta)^2}{4K} \right] h(\zeta, P, K) \quad (2.134)$$

where $P = p + K\zeta$. The constraint $C_0 = 0$ becomes now an ordinary differential equation on h ,

$$\left[\partial_\zeta^2 + 2(P - 2K\zeta) \partial_\zeta - 2K(\nu + 1) \right] h(\zeta, P, K) \equiv 0 \quad (2.135)$$

For $\nu = -1$, the solutions are

$$h(\zeta, P, K) = h_1(P, K) + h_2(P, K) K^{-1/2} e^{-\frac{P^2}{2K}} \operatorname{erfi} \left(-\frac{P - 2K\zeta}{\sqrt{2K}} \right) \quad (2.136)$$

Only the ζ -independent part $h_1(P, K)$ obeys also (2.114a). Changing variables again by setting $K = x^2/2 = y/2$ and $P = -xu = -x_0$, it is easy to check that the quasi-conformal action on $f(\zeta, \tilde{\zeta}, \sigma)$ at $k = 0$ gives precisely the deformed minimal representation (2.111) acting on $h_1(P, K)$ at $\nu = -1$. We conclude that the deformed minimal representation at $\nu = -1$ can be embedded inside the principal series representation at $k = 0$ by

$$f(\zeta, \tilde{\zeta}, \sigma) = \int dp dK \exp \left(-iK\sigma - ip\tilde{\zeta} - \frac{(p - K\zeta)^2}{4K} \right) h(p + K\zeta, K) . \quad (2.137)$$

The formula (2.137) can also be viewed as the matrix element

$$f(\zeta, \tilde{\zeta}, \sigma) = \langle f_P | e^{-\sigma E - \zeta E_q + \tilde{\zeta} E_p} | h \rangle \quad (2.138)$$

where f_P is the P -covariant vector in the deformed minimal representation (2.115).

For $\nu = 1$, similar arguments show that the deformed minimal representation can be embedded inside the principal series representation at $k = 0$ via

$$f(\zeta, \tilde{\zeta}, \sigma) = \int dp dK \exp \left(-iK\sigma - ip\tilde{\zeta} + \frac{(p - K\zeta)^2}{4K} \right) h(p + K\zeta, K) . \quad (2.139)$$

For other values of ν , the solution of (2.135) involves Hermite and hypergeometric functions,

$$h = h_1(P, K) H_{-\frac{\nu+1}{2}} \left(-\frac{P - 2K\zeta}{\sqrt{2K}} \right) + h_2(P, K) {}_1F_1 \left(\frac{1+\nu}{4}, \frac{1}{2}; \frac{(P - 2K\zeta)^2}{2K} \right) \quad (2.140)$$

It is worth noting that the formula (2.137) admits a simple generalization to all quaternionic groups, as we shall see for $G_{2(2)}$ in (3.148) below.

2.6 The minimal representation as a quantized quasi-conformal action

We now explain how the minimal representation can be viewed as the quantization of the quasi-conformal action of G , or equivalently how the quaternionic discrete series arises as a semi-classical limit of the minimal representation.

2.6.1 Lifting the quasi-conformal action to the hyperkähler cone

As a first step, it is useful to “deprojectivize” the quasi-conformal action, i.e. lift it to an action on the hyperkähler cone \mathcal{S} . For this purpose, we introduce an extra variable t and interpret the k -dependent quasi-conformal action as an action of functions of four variables $\hat{f}(t, \xi, \tilde{\xi}, \alpha) = e^{-kt} f(\xi, \tilde{\xi}, \alpha)$. We then implement the change of variables found in [36],

$$v^b = e^{2t} , \quad v^0 = \xi e^{2t} , \quad w_0 = \frac{i}{2} \tilde{\xi} , \quad w_b = \frac{1}{4i} (\alpha + \xi \tilde{\xi}) \quad (2.141)$$

The coordinates v^b, v^0, w_b, w_0 are complex coordinates on the Swann space \mathcal{S} , such that the holomorphic symplectic form takes the Darboux form

$$\Omega = dw_b \wedge dv^b + dw_0 \wedge dv^0 \quad (2.142)$$

The quasi-conformal action on $\mathcal{O}(-k)$ over \mathcal{Z} now corresponds to the holomorphic action of G on \mathcal{S} , restricted to the subspace of homogeneous functions of degree $-k$ under the rescaling

$$v^I \rightarrow \mu^2 v^I , \quad w_I \rightarrow w_I \quad (2.143)$$

The holomorphic vector fields generating the action of G are given by¹⁸

$$\begin{aligned}
E_p &= \frac{i}{2} \partial_{w_0} , \quad E_q = w_0 \partial_{w_b} - v^b \partial_{v^0} , \quad E = -\frac{i}{4} \partial_{w_b} \\
H &= 2v^b \partial_{v^b} - 2w_b \partial_{w_b} + v^0 \partial_{v^0} - w_0 \partial_{w_0} , \\
J &= -i \left(w_0^2 + \frac{(v^0)^2}{v^b} \right) \partial_{w_b} + 2iv^b w_0 \partial_{v^0} + \frac{iv^0}{v^b} \partial_{w_0} \\
F_p &= -4iv^b w_0 \partial_{v^b} + \left(4iw_b w_0 - \frac{iv^{03}}{2v^{b3}} \right) \partial_{w_b} - 2i(2v^b w_b + v^0 w_0) \partial_{v^0} + \left(\frac{3iv^{02}}{4v^{b2}} + iw_0^2 \right) \partial_{w_0} \\
F_q &= -2v^0 \partial_{v^b} - \left(2w_0^3 + \frac{3(v^0)^2 w_0}{2(v^b)^2} \right) \partial_{w_b} + \left(6v^b w_0^2 - \frac{3(v^0)^2}{2v^b} \right) \partial_{v^0} + \left(2w_b + \frac{3v^0 w_0}{v^b} \right) \partial_{w_0} \\
F &= 4i(2v^b w_b + v^0 w_0) \partial_{v^b} + \frac{i \left(16(w_0^4 - 4w_b^2)(v^b)^4 + 24v^{02} w_0^2 v^{b2} - 3v^{04} \right)}{16(v^b)^4} \partial_{w_b} \\
&\quad + i \left(-4v^b w_0^3 + \frac{3(v^0)^2 w_0}{v^b} + 4v^0 w_b \right) \partial_{v^0} - \frac{i \left(16w_b w_0 v^{b3} + 12v^0 w_0^2 (v^b)^2 + (v^0)^3 \right)}{4(v^b)^3} \partial_{w_0}
\end{aligned} \tag{2.144}$$

Being tri-holomorphic isometries, these vector fields in particular preserve the holomorphic symplectic form Ω . They can be represented by holomorphic moment maps \underline{X} , such that the contraction $\iota_X \Omega = d\underline{X}$:

$$\begin{aligned}
\underline{E}_p &= \frac{i}{2} v^0 , \quad \underline{E}_q = v^b w_0 , \quad \underline{E} = -\frac{i}{4} v^b , \quad \underline{H} = -2v^b w_b - v^0 w_0 , \quad \underline{J} = -w_0^2 - \frac{i}{4} \frac{(v^0)^2}{(v^b)^2} \\
\underline{F}_p &= \frac{i}{4} \left(\frac{(v^0)^3}{(v^b)^2} + 4(w_0)^2 v^0 + 16v^b w_b w_0 \right) , \quad \underline{F}_q = -2v^b (w_0)^3 + \frac{3(v^0)^2 w_0}{2v^b} + 2v^0 w_b \\
\underline{F} &= \frac{i}{16(v^b)^3} \left(16(w_0^4 - 4w_b^2)(v^b)^4 - 64v^0 w_b w_0 (v^b)^3 - 24(v^0 w_0 v^b)^2 + (v^0)^4 \right) \tag{2.145}
\end{aligned}$$

Returning to the variables $t, \xi, \tilde{\xi}, \alpha$, the holomorphic moment maps become,

$$\begin{aligned}
\underline{E}_p &= \frac{i}{2} e^{2t} \xi , \quad \underline{E}_q = \frac{i}{2} e^{2t} \tilde{\xi} , \quad \underline{E} = -\frac{i}{4} e^{2t} , \quad \underline{H} = \frac{i}{2} e^{2t} \alpha , \\
\underline{F}_p &= \frac{i}{4} e^{2t} \left[\xi(\xi^2 + \tilde{\xi}^2) + 2\alpha \tilde{\xi} \right] , \quad \underline{F}_q = \frac{i}{4} e^{2t} \left[\tilde{\xi}(\xi^2 + \tilde{\xi}^2) - 2\alpha \xi \right] , \\
\underline{F} &= \frac{i}{16} e^{2t} \left[4\alpha^2 + (\xi^2 + \tilde{\xi}^2)^2 \right] , \quad \underline{J} = \frac{i}{4} e^{2t} (\xi^2 + \tilde{\xi}^2)
\end{aligned} \tag{2.146}$$

Arranging these holomorphic moment maps into an element Q of \mathfrak{g}^* , one may check that $Q^2 = 0$ in our matrix representation, i.e. that Q is valued in the minimal coadjoint orbit. In particular, the holomorphic moment maps satisfy classical versions of

¹⁸Note that only real combinations $E + \bar{E}$ are actual isometries of the hyperkähler metric.

the identities (2.104),

$$\underline{H}^2 + 4\underline{E}\underline{F} + \underline{J}^2 = 0 \quad (2.147a)$$

$$\underline{E}_p^2 + \underline{E}_q^2 + 4\underline{J}\underline{E} = 0 \quad (2.147b)$$

$$C_2(\underline{J}) + \frac{1}{9}\underline{S}^2 = 0 \quad (2.147c)$$

$$\underline{C}_2 = \underline{C}_3 = 0 \quad (2.147d)$$

$$\underline{J}_{\frac{1}{2}, \frac{3}{2}} \underline{J}_{\frac{1}{2}, -\frac{3}{2}} - \frac{4i}{3} \underline{S} \underline{J}_+ = 0 \quad (2.147e)$$

The holomorphic moment maps of the compact generators are given by

$$\underline{S} = \frac{3i}{16} e^{2t} \left(1 + \xi^2 + \tilde{\xi}^2 + \alpha^2 + \frac{1}{4}(\xi^2 + \tilde{\xi})^2 \right) \quad (2.148a)$$

$$\underline{J}_3 = \frac{i}{16} e^{2t} \left(1 - 3\xi^3 - 3\tilde{\xi}^2 + \alpha^2 + \frac{1}{4}(\xi^2 + \tilde{\xi}^2)^2 \right) \quad (2.148b)$$

$$\underline{J}_\pm = \frac{i}{8\sqrt{2}} e^{2t} \left(\xi \mp i\tilde{\xi} \right) \left(\xi^2 + \tilde{\xi}^2 \pm 2i\alpha - 2 \right) \quad (2.148c)$$

$$\underline{J}_{\pm\frac{1}{2}, \pm\frac{3}{2}} = \frac{1}{16} e^{2t} \left[(\xi^2 + \tilde{\xi}^2)^2 + 4\alpha(\alpha \pm 2i) - 4 \right] \quad (2.148d)$$

$$\underline{J}_{\pm\frac{1}{2}, \mp\frac{3}{2}} = -\frac{i}{8\sqrt{2}} e^{2t} \left(\xi \mp i\tilde{\xi} \right) \left(\xi^2 + \tilde{\xi}^2 \pm 2i\alpha + 2 \right) \quad (2.148e)$$

It is interesting to note that the K -type (2.79) may be rewritten in terms of these moment maps, up to an overall numerical factor, as

$$f_{j,s} = \left(\frac{\underline{J}_{\frac{1}{2}, \frac{3}{2}}}{\underline{S}} \right)^{j+\frac{1}{3}s} \left(\frac{\underline{J}_{\frac{1}{2}, -\frac{3}{2}}}{\underline{S}} \right)^{j-\frac{1}{3}s} \underline{S}^{-\frac{1}{2}k} \quad (2.149)$$

The covariance of (2.149) under K is then easy to see, using the identities (2.147) obeyed by the holomorphic moment maps, and the fact that G acts by Poisson brackets, $Xf = \{X, f\}$.

Moreover, the vanishing locus of the holomorphic moment maps associated to the compact generators S, J_3, J_\pm consists of two branches

$$\xi = \pm i\tilde{\xi}, \quad \alpha = \pm i \quad (2.150)$$

or equivalently, in terms of the variables on the hyperkähler cone,

$$w_b = \pm \frac{1}{4} \left(1 - \frac{(v^0)^2}{(v^b)^2} \right), \quad w_0 = \pm \frac{v^0}{2v^b} \quad (2.151)$$

These relations define two lagrangian cones \mathcal{C}_\pm , with generating functions S_\pm : they may be rewritten as

$$w_b = \partial_{v^b} S_\pm, \quad w_0 = \partial_{v^0} S_\pm, \quad S_\pm(v^b, v^0) = \pm \frac{1}{4} \left(v^b + \frac{(v^0)^2}{v^b} \right) \quad (2.152)$$

By construction, \mathcal{C}_\pm are invariant under the holomorphic action of K , since the Poisson brackets with the constraints vanish on the constraint locus. As we shall see momentarily, S_- describes the semi-classical limit of the lowest K -type (2.119) of the minimal representation.

2.6.2 The classical limit of the minimal representation

We now return to the presentation (2.103) of the minimal representation acting on functions of y, x_0 . The form (2.119) of the spherical vector suggests that a semi-classical limit exists as y, x_0 are scaled simultaneously to infinity, if one restricts to wave functions of the form

$$f(y, x_0) = \exp [S(y, x_0)] \quad , \quad S(y, x_0) = y \hat{S}(\hat{x}_0) + \mathcal{O}(1) \quad (2.153)$$

where $\hat{x}_0 = x_0/y$ is kept fixed in the limit $y \rightarrow \infty$. Indeed, it is easy to check that, to leading order in this limit, the action of the infinitesimal generators on f produces

$$X \cdot f = y \underline{X} \left(\hat{S}, \partial_{\hat{0}} \hat{S}, \hat{x}^0 \right) f + \mathcal{O}(y), \quad (2.154)$$

for some (so far unspecified) function \underline{X} . Changing variables to

$$p_0 = \partial_{x_0} S(y, x_0) = \partial_{\hat{x}_0} \hat{S} \quad , \quad p_y = \partial_y S(y, x_0) = \hat{S} - \hat{x}_0 \partial_{\hat{x}_0} \hat{S} \quad , \quad (2.155)$$

identifies the function $\underline{X}(y, x_0, p_y, p_0)$ as the leading differential symbol of the differential operator X in the semi-classical limit (2.153). Further setting

$$v^b = -2y \quad , \quad v^0 = 2x_0 \quad , \quad w_b = \frac{1}{2} p_y \quad , \quad w_0 = -\frac{1}{2} p_0 \quad , \quad (2.156)$$

identifies \underline{X} as the holomorphic moment map (2.145) associated to the tri-holomorphic action of G on its hyperkähler cone \mathcal{S} . Thus, we conclude that the minimal representation can be viewed as the quantization of the holomorphic symplectic manifold \mathcal{S} . Furthermore, the semi-classical limit of the spherical vector (2.119) is given by the generating function S_- of the Lagrangian cone \mathcal{S}_- defined in (2.152).

One could also have given a real version of this construction, lifting the action of G on $P \backslash G$ to a real symplectic manifold by adding a single real coordinate. In that case we would say that the minimal representation arises by quantizing the real symplectic structure. Indeed, the real manifold so obtained is at least locally isomorphic to the minimal coadjoint orbit of G , so this makes contact with one of the standard ways of thinking about the minimal representation.

3. $G_{2(2)}$

In this section, we describe the geometry of the quaternionic-Kähler space $SO(4) \backslash G_{2(2)}$, and various associated unitary representations of $G = G_{2(2)}$. We will be somewhat briefer in this section since many of the constructions are parallel to ones we described for $G = SU(2, 1)$ above.

3.1 Some group theory

It is convenient to represent the Lie algebra \mathfrak{g} of $G = G_{2(2)}$ by the 7-dimensional matrix representation described in [56] (after some relabelings and change of normalization)

$$\begin{pmatrix} \underline{Y}_0 & -\underline{Y}_+ & 0 & -\sqrt{\frac{2}{3}}\underline{E}_{q1} & \sqrt{\frac{2}{3}}\underline{E}_{p1} & -\sqrt{2}\underline{E}_{q0} & \sqrt{2}\underline{E}_{p0} \\ \underline{Y}_- & 0 & -\underline{Y}_+ & \frac{2}{\sqrt{3}}\underline{F}_{p1} & \frac{2}{\sqrt{3}}\underline{E}_{q1} & \frac{2}{\sqrt{3}}\underline{F}_{q1} & -\frac{2}{\sqrt{3}}\underline{E}_{p1} \\ 0 & \underline{Y}_- & -\underline{Y}_0 & \sqrt{2}\underline{E}_{p0} & \sqrt{2}\underline{E}_{q0} & -\sqrt{\frac{2}{3}}\underline{F}_{p1} & -\sqrt{\frac{2}{3}}\underline{E}_{q1} \\ -\sqrt{\frac{2}{3}}\underline{E}_{q1} & \frac{2}{\sqrt{3}}\underline{E}_{p1} & \sqrt{2}\underline{E}_{p0} & \underline{H} + \frac{1}{2}\underline{Y}_0 & -\underline{E} & -\frac{1}{\sqrt{2}}\underline{Y}_+ & 0 \\ \sqrt{\frac{2}{3}}\underline{F}_{p1} & \frac{2}{\sqrt{3}}\underline{F}_{q1} & \sqrt{2}\underline{F}_{q0} & -\underline{F} & -\underline{H} + \frac{1}{2}\underline{Y}_0 & 0 & -\frac{1}{\sqrt{2}}\underline{Y}_+ \\ -\sqrt{2}\underline{E}_{q0} & \frac{2}{\sqrt{3}}\underline{E}_{q1} & -\sqrt{\frac{2}{3}}\underline{E}_{p1} & \frac{1}{\sqrt{2}}\underline{Y}_- & 0 & \underline{H} - \frac{1}{2}\underline{Y}_0 & -\underline{E} \\ \sqrt{2}\underline{F}_{p0} & -\frac{2}{\sqrt{3}}\underline{F}_{p1} & -\sqrt{\frac{2}{3}}\underline{F}_{q1} & 0 & \frac{1}{\sqrt{2}}\underline{Y}_- & -\underline{F} & -\frac{1}{2}\underline{Y}_0 - \underline{H} \end{pmatrix} \equiv \underline{X}_i X_i \quad (3.1)$$

where, as in (2.2), \underline{X}_i are real coordinates dual to the generators, to be represented by anti-hermitian operators in a given unitary representation. These matrices preserve the signature $(4^+, 3^-)$ metric

$$\eta_{ij} dx_i dx_j = -2dx_1 dx_3 + dx_2^2 + 2dx_4 dx_7 - 2dx_5 dx_6 \quad (3.2)$$

and the three-form

$$dx_{123} - dx_{247} + dx_{256} + \sqrt{2}dx_{167} + \sqrt{2}dx_{345} , \quad (3.3)$$

thus providing an embedding of $G_{2(2)}$ inside $SO(3, 4)$. In addition to the universal commutation relations

$$\begin{aligned} [E, F] &= H, & [H, E] &= 2E, & [H, F] &= -2F, \\ [E_{p^I}, E_{q^J}] &= -2\delta_J^I E, & [F_{p^I}, F_{q^J}] &= 2\delta_J^I F, \\ [E_{p^I}, E_{p^J}] &= 0, & [F_{p^I}, F_{p^J}] &= 0, & [E_{qI}, E_{qJ}] &= 0, & [F_{qI}, F_{qJ}] &= 0, \\ [H, E_{p^I}] &= E_{p^I}, & [H, F_{p^I}] &= -F_{p^I}, & [H, E_{qI}] &= E_{qI}, & [H, F_{qI}] &= -F_{qI}, \\ [F, E_{p^I}] &= -F_{qI}, & [E, F_{qI}] &= -F_{p^I}, & [F, E_{qI}] &= F_{p^I}, & [E, F_{p^I}] &= E_{qI}, \end{aligned} \quad (3.4)$$

we have the $SL(2, \mathbb{R})$ algebra

$$[Y_0, Y_{\pm}] = \pm Y_{\pm} , \quad [Y_-, Y_+] = Y_0 \quad (3.5)$$

under which E, F are singlets

$$[Y_0, E] = [Y_{\pm}, E] = [Y_0, F] = [Y_{\pm}, F] = 0 \quad (3.6)$$

and $E_{p,q}$ and $F_{p,q}$ transform as a spin $3/2$,

$$\left[Y_0, \begin{pmatrix} E_{p^0} \\ E_{p^1} \\ E_{q_1} \\ E_{q_0} \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 3E_{p^0} \\ E_{p^1} \\ -E_{q_1} \\ -3E_{q_0} \end{pmatrix} , \quad \left[Y_0, \begin{pmatrix} F_{p^0} \\ F_{p^1} \\ F_{q_1} \\ F_{q_0} \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} -3F_{p^0} \\ -F_{p^1} \\ F_{q_1} \\ 3F_{q_0} \end{pmatrix} \quad (3.7a)$$

$$\left[Y_+, \begin{pmatrix} E_{p^0} \\ E_{p^1} \\ E_{q_1} \\ E_{q_0} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \sqrt{\frac{3}{2}}E_{p^0} \\ -\sqrt{2}E_{p^1} \\ -\sqrt{\frac{3}{2}}E_{q_1} \end{pmatrix} , \quad \left[Y_+, \begin{pmatrix} F_{p^0} \\ F_{p^1} \\ F_{q_1} \\ F_{q_0} \end{pmatrix} \right] = \begin{pmatrix} -\sqrt{\frac{3}{2}}F_{p^1} \\ \sqrt{2}F_{q_1} \\ \sqrt{\frac{3}{2}}F_{q_0} \\ 0 \end{pmatrix} \quad (3.7b)$$

$$\left[Y_-, \begin{pmatrix} E_{p^0} \\ E_{p^1} \\ E_{q_1} \\ E_{q_0} \end{pmatrix} \right] = \begin{pmatrix} -\sqrt{\frac{3}{2}}E_{p^1} \\ \sqrt{2}E_{q_1} \\ \sqrt{\frac{3}{2}}E_{q_0} \\ 0 \end{pmatrix} , \quad \left[Y_-, \begin{pmatrix} F_{p^0} \\ F_{p^1} \\ F_{q_1} \\ F_{q_0} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \sqrt{\frac{3}{2}}F_{p^0} \\ -\sqrt{2}F_{p^1} \\ -\sqrt{\frac{3}{2}}F_{q_1} \end{pmatrix} \quad (3.7c)$$

Moreover, the commutation between positive and negative roots give

$$[E_{p^0}, F_{p^0}] = H + 2Y_0 , \quad [E_{q_0}, F_{q_0}] = H - 2Y_0 \quad (3.8a)$$

$$[E_{p^1}, F_{p^1}] = \frac{1}{3}(3H + 2Y_0) , \quad [E_{q_1}, F_{q_1}] = \frac{1}{3}(3H - 2Y_0), \quad (3.8b)$$

$$[E_{p^1}, F_{q_1}] = -\frac{4\sqrt{2}}{3}Y_+ , \quad [E_{q_1}, F_{p^1}] = \frac{4\sqrt{2}}{3}Y_- \quad (3.8c)$$

The quadratic Casimir is

$$\begin{aligned} C_2 = & \frac{1}{4} (H^2 + 2EF + 2FE) + \frac{1}{3} (Y_0^2 - Y_+Y_- - Y_-Y_+) \\ & + \frac{1}{4} \sum_{I=0,1} (E_{p^I} F_{p^I} + F_{p^I} E_{p^I} + E_{q_I} F_{q_I} + F_{q_I} E_{q_I}) \end{aligned} \quad (3.9)$$

normalized so that

$$C_2(adj) = 4 , \quad C_2(\mathbf{7}) = 2 \quad (3.10)$$

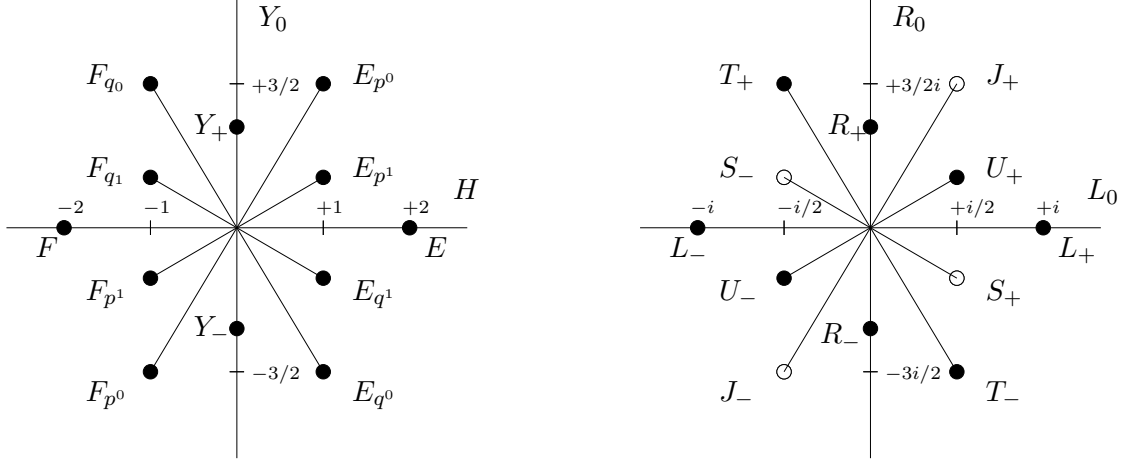


Figure 4: Root diagram of $G_{2(2)}$ with respect to the split Cartan torus H, Y_0 (left) and the compact Cartan torus L_0, R_0 (right). The compact (resp. non-compact) roots are indicated by a white (resp. black) dot. The long roots generate $SL(3, \mathbb{R})$ (left) and $SU(2, 1)$ (right) subgroups, respectively.

There is also a degree 6 Casimir, corresponding to the trace of the sixth power of the matrix (3.1), which we shall not attempt to write.

This basis is adapted to the maximal subgroup $SL(2, \mathbb{R})_{\text{short}} \times SL(2, \mathbb{R})_{\text{long}}$, where the first factor is generated by $\{H, E, F\}$ while the second is generated by $\{Y_-, Y_0, Y_+\}$. The Cartan generators H, Y_0 are non-compact, with spectrum

$$\text{Spec}(Y_0) = \{0, \pm 1/2, \pm 1, \pm 3/2\}, \quad \text{Spec}(H) = \{0, \pm 1, \pm 2\} \quad (3.11)$$

The non-compact generator H gives rise to the “real non-compact 5-grading”

$$F|_{-2} \oplus \{F_{p^I}, F_{q_I}\}|_{-1} \oplus \{H, Y_0, Y_{\pm}\}|_0 \oplus \{E_{p^I}, E_{q_I}\}|_1 \oplus E|_2 \quad (3.12)$$

Define a parabolic subgroup $P = LN$ with Levi $L = \mathbb{R} \times SL(2, \mathbb{R})$ generated by $\{H, Y_0, Y_+, Y_-\}$ and unipotent radical N generated by $\{F_{q_0}, F_{q_1}, F_{p^0}, F_{p^1}, F\}$, corresponding to the spaces with zero and negative grade in the decomposition (3.12). We call P the Heisenberg parabolic subgroup. In section 3.7, we will also be interested in the parabolic subgroup $P_3 = L_3 N_3$ associated to the 7-grading induced by Y_0 , with Levi $L_3 = \mathbb{R} \times SL(2, \mathbb{R}) = \{Y_0, E, F, H\}$ and unipotent radical $N_3 = \{E_{q_0}, E_{q_1}, F_{p^0}, F_{p^1}, Y_-\}$, nilpotent of degree 3.

Now we introduce a different basis adapted to the maximal compact subgroup

$SU(2) \times SU(2)$. We first go to a compact basis for the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ group:

$$L_{\pm} = -\frac{1}{2\sqrt{2}}(E + F \pm iH) , \quad L_0 = \frac{1}{2}(F - E) \quad (3.13a)$$

$$R_0 = \frac{1}{\sqrt{2}}(Y_+ + Y_-) , \quad R_{\pm} = \frac{1}{2}(Y_+ - Y_- \mp i\sqrt{2}Y_0) \quad (3.13b)$$

such that

$$[L_0, L_{\pm}] = \pm iL_{\pm} , \quad [L_+, L_-] = -iL_0 \quad (3.14a)$$

$$[R_0, R_{\pm}] = \pm iR_{\pm} , \quad [R_+, R_-] = -iR_0 \quad (3.14b)$$

Moreover, we define the eigenmodes

$$\begin{aligned} K_{\frac{1}{2}, \frac{3}{2}} &= \frac{\sqrt{2}}{4} \left[-(E_{p^0} - iE_{q_0}) - i\sqrt{3}(E_{p^1} - iE_{q_1}) + (F_{p^0} - iF_{q_0}) + i\sqrt{3}(F_{p^1} - iF_{q_1}) \right] \\ K_{\frac{1}{2}, \frac{1}{2}} &= \frac{\sqrt{2}}{4} \left[\sqrt{3}(E_{p^0} + iE_{q_0}) + i(E_{p^1} + iE_{q_1}) + \sqrt{3}(F_{p^0} + iF_{q_0}) + i(F_{p^1} + iF_{q_1}) \right] \\ K_{\frac{1}{2}, -\frac{1}{2}} &= \frac{\sqrt{2}}{4} \left[\sqrt{3}(E_{p^0} - iE_{q_0}) - i(E_{p^1} - iE_{q_1}) - \sqrt{3}(F_{p^0} - iF_{q_0}) + i(F_{p^1} - iF_{q_1}) \right] \\ K_{\frac{1}{2}, -\frac{3}{2}} &= \frac{\sqrt{2}}{4} \left[-(E_{p^0} + iE_{q_0}) + i\sqrt{3}(E_{p^1} + iE_{q_1}) - (F_{p^0} - iF_{q_0}) + i\sqrt{3}(F_{p^1} + iF_{q_1}) \right] \end{aligned} \quad (3.15)$$

where the eigenvalues under of $(-iL_0, -iR_0)$ are indicated in subscript. Note that the hermiticity conditions are now

$$L_{\pm}^{\dagger} = -L_{\mp} , \quad R_{\pm}^{\dagger} = -R_{\mp} , \quad K_{-l_0, -r_0} = -K_{l_0, r_0}^{\dagger} \quad (3.16)$$

In this basis, the quadratic Casimir becomes

$$\begin{aligned} C_2 &= -\frac{1}{3}(R_0^2 - R_-R_+ - R_+R_-) - (L_0^2 - L_-L_+ - L_+L_-) \\ &\quad -\frac{1}{8} \left(K_{\frac{1}{2}, \frac{3}{2}} K_{-\frac{1}{2}, -\frac{3}{2}} + K_{-\frac{1}{2}, -\frac{3}{2}} K_{\frac{1}{2}, \frac{3}{2}} \right) + \frac{1}{8} \left(K_{-\frac{1}{2}, \frac{3}{2}} K_{\frac{1}{2}, -\frac{3}{2}} + K_{\frac{1}{2}, -\frac{3}{2}} K_{-\frac{1}{2}, \frac{3}{2}} \right) \\ &\quad -\frac{1}{8} \left(K_{\frac{1}{2}, -\frac{1}{2}} K_{-\frac{1}{2}, \frac{1}{2}} + K_{-\frac{1}{2}, \frac{1}{2}} K_{\frac{1}{2}, -\frac{1}{2}} \right) + \frac{1}{8} \left(K_{\frac{1}{2}, \frac{1}{2}} K_{-\frac{1}{2}, -\frac{1}{2}} + K_{-\frac{1}{2}, -\frac{1}{2}} K_{\frac{1}{2}, \frac{1}{2}} \right) \end{aligned} \quad (3.17)$$

This shows in particular that the roots $K_{\pm\frac{1}{2}, \pm\frac{3}{2}}$ and $K_{\pm\frac{1}{2}, \mp\frac{1}{2}}$ are compact, as indicated on the diagram.

The compact Cartan generator L_0 gives rise to the “non-compact holomorphic 5-grading”

$$L_- |_{-2i} \oplus \{K_{-\frac{1}{2}, \pm\frac{1}{2}}, K_{-\frac{1}{2}, \pm\frac{3}{2}}\} |_{-i} \oplus \{L_0, R_0, R_{\pm}\} |_0 \oplus \{K_{\frac{1}{2}, \pm\frac{1}{2}}, K_{\frac{1}{2}, \pm\frac{3}{2}}\} |_i \oplus L_+ |_{2i} \quad (3.18)$$

Now, we perform a $\pi/3$ rotation of the root diagram, and define

$$J_3 = \frac{1}{2}(L_0 + R_0) = \frac{1}{4}(F - E) + \frac{1}{2\sqrt{2}}(Y_+ + Y_-) \quad (3.19a)$$

$$S_3 = \frac{1}{2}(3L_0 - R_0) = \frac{3}{4}(F - E) - \frac{1}{2\sqrt{2}}(Y_+ + Y_-) \quad (3.19b)$$

as the new Cartan algebra. The new eigenmodes are now

$$J_- = \frac{1}{2}K_{-\frac{1}{2}, -\frac{3}{2}}, \quad J_+ = \frac{1}{2}K_{\frac{1}{2}, \frac{3}{2}}, \quad S_- = \frac{3}{2}K_{-\frac{1}{2}, \frac{1}{2}}, \quad S_+ = \frac{3}{2}K_{\frac{1}{2}, -\frac{1}{2}} \quad (3.20)$$

$$J_{\frac{1}{2}, -\frac{3}{2}} = K_{-\frac{1}{2}, \frac{3}{2}}, \quad J_{\frac{1}{2}, \frac{3}{2}} = 2\sqrt{2}L_+, \quad J_{\frac{1}{2}, -\frac{1}{2}} = 2\sqrt{\frac{2}{3}}R_+, \quad J_{\frac{1}{2}, \frac{1}{2}} = K_{\frac{1}{2}, \frac{1}{2}} \quad (3.20)$$

together with their hermitian conjugates. They satisfy the $SU(2) \times SU(2)$ algebra

$$[J_3, J_{\pm}] = \pm iJ_{\pm}, \quad [J_+, J_-] = 2iJ_3 \quad (3.21a)$$

$$[S_3, S_{\pm}] = \pm iS_{\pm}, \quad [S_+, S_-] = 2iS_3 \quad (3.21b)$$

The subscript on J now denotes the eigenvalues under $(-iJ_3, -iS_3)$. In terms of the non-compact basis, the compact generators are as usual differences between positive and negative roots,

$$F_{p^0} - E_{p^0} = \frac{1}{2}(S_+ + S_- + J_+ + J_-) \quad (3.22a)$$

$$F_{p^1} - E_{p^1} = \frac{i}{2\sqrt{3}}(S_+ - S_- - 3J_+ + 3J_-) \quad (3.22b)$$

$$F_{q_0} - E_{q_0} = \frac{i}{2}(S_+ - S_- + J_+ - J_-) \quad (3.22c)$$

$$F_{q_1} - E_{q_1} = \frac{1}{2\sqrt{3}}(-S_+ - S_- + 3J_+ + 3J_-) \quad (3.22d)$$

$$F - E = S_3 + J_3 \quad (3.22e)$$

$$Y_+ + Y_- = \frac{1}{\sqrt{2}}(3J_3 - S_3) \quad (3.22f)$$

The matrix representation adapted to this compact basis is obtained from (3.1) by a Cayley rotation

$$C\eta C^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C = e^{\frac{\pi\sqrt{2}}{4}(R_+ + R_-)} = \begin{pmatrix} 0 & 0 & 0 & \frac{i}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{\sqrt{2}} & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & \frac{1}{2} & \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{\sqrt{2}} & -\frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{i}{2} \end{pmatrix} \quad (3.23)$$

The generators in the compact basis then have the matrix representation

$$\begin{pmatrix} \frac{1}{2}i(\underline{J}_3 + \underline{S}_3) & -\underline{S}_+ & -\frac{2i\underline{J}_{\frac{1}{2},-\frac{1}{2}}}{\sqrt{3}} & 2i\sqrt{\frac{2}{3}}\underline{J}_{\frac{1}{2},\frac{1}{2}} & -2\underline{J}_{\frac{1}{2},\frac{3}{2}} & \underline{J}_+ & 0 \\ \underline{S}_- & \frac{1}{2}i(\underline{J}_3 - \underline{S}_3) & -2\underline{J}_{\frac{1}{2},-\frac{3}{2}} & -2\sqrt{\frac{2}{3}}\underline{J}_{\frac{1}{2},-\frac{1}{2}} & -\frac{2\underline{J}_{\frac{1}{2},\frac{1}{2}}}{\sqrt{3}} & 0 & -\underline{J}_+ \\ \frac{2i\underline{J}_{-\frac{1}{2},\frac{1}{2}}}{\sqrt{3}} & -2\underline{J}_{-\frac{1}{2},\frac{3}{2}} & i\underline{S}_3 & -i\sqrt{2}\underline{S}_+ & 0 & -\frac{2\underline{J}_{\frac{1}{2},\frac{1}{2}}}{\sqrt{3}} & -2\underline{J}_{\frac{1}{2},\frac{3}{2}} \\ -2i\sqrt{\frac{2}{3}}\underline{J}_{-\frac{1}{2},-\frac{1}{2}} & -2\sqrt{\frac{2}{3}}\underline{J}_{-\frac{1}{2},\frac{1}{2}} & -i\sqrt{2}\underline{S}_- & 0 & i\sqrt{2}\underline{S}_+ & -2\sqrt{\frac{2}{3}}\underline{J}_{\frac{1}{2},-\frac{1}{2}} & 2i\sqrt{\frac{2}{3}}\underline{J}_{\frac{1}{2},\frac{1}{2}} \\ -2\underline{J}_{-\frac{1}{2},-\frac{3}{2}} & -\frac{2\underline{J}_{\frac{1}{2},-\frac{1}{2}}}{\sqrt{3}} & 0 & i\sqrt{2}\underline{S}_- & -i\underline{S}_3 & -2\underline{J}_{\frac{1}{2},-\frac{3}{2}} & -\frac{2i\underline{J}_{\frac{1}{2},-\frac{1}{2}}}{\sqrt{3}} \\ -\underline{J}_- & 0 & -\frac{2\underline{J}_{-\frac{1}{2},-\frac{1}{2}}}{\sqrt{3}} & -2\sqrt{\frac{2}{3}}\underline{J}_{-\frac{1}{2},\frac{1}{2}} & -2\underline{J}_{-\frac{1}{2},\frac{3}{2}} & -\frac{1}{2}i(\underline{J}_3 - \underline{S}_3) & \underline{S}_+ \\ 0 & \underline{J}_- & -2\underline{J}_{-\frac{1}{2},-\frac{3}{2}} & -2i\sqrt{\frac{2}{3}}\underline{J}_{-\frac{1}{2},-\frac{1}{2}} & \frac{2i\underline{J}_{-\frac{1}{2},\frac{1}{2}}}{\sqrt{3}} & -\underline{S}_- & -\frac{1}{2}i(\underline{J}_3 + \underline{S}_3) \end{pmatrix} \quad (3.24)$$

The advantage of this description is that the Harish-Chandra decomposition with respect to J_3 is simply a generalized LU decomposition in $2+3+2$ blocks.

The quadratic Casimir in the compact basis becomes

$$\begin{aligned} C_2 = & -\frac{1}{3}(S_3^2 + \frac{1}{2}S_-S_+ + \frac{1}{2}S_+S_-) - (J_3^2 + \frac{1}{2}J_-J_+ + \frac{1}{2}J_+J_-) \\ & + \frac{1}{8} \left(J_{\frac{1}{2},\frac{3}{2}}J_{-\frac{1}{2},-\frac{3}{2}} + J_{-\frac{1}{2},-\frac{3}{2}}J_{\frac{1}{2},\frac{3}{2}} \right) + \frac{1}{8} \left(J_{-\frac{1}{2},\frac{3}{2}}J_{\frac{1}{2},-\frac{3}{2}} + J_{\frac{1}{2},-\frac{3}{2}}J_{-\frac{1}{2},\frac{3}{2}} \right) \\ & + \frac{1}{8} \left(J_{\frac{1}{2},-\frac{1}{2}}J_{-\frac{1}{2},\frac{1}{2}} + J_{-\frac{1}{2},\frac{1}{2}}J_{\frac{1}{2},-\frac{1}{2}} \right) + \frac{1}{8} \left(J_{\frac{1}{2},\frac{1}{2}}J_{-\frac{1}{2},-\frac{1}{2}} + J_{-\frac{1}{2},-\frac{1}{2}}J_{\frac{1}{2},\frac{1}{2}} \right) \end{aligned} \quad (3.25)$$

which makes it clear that the compact roots are J_{\pm}, S_{\pm} .

The generator J_3 now gives rise to the “compact 5-grading”

$$J_-|_{-2i} \oplus \{T_-, R_-, U_-, L_-\}|_{-i} \oplus \{J_3, S_3, S_+, S_-\}|_0 \oplus \{T_+, R_+, U_+, L_+\}|_i \oplus J_+|_{2i} \quad (3.26)$$

where we denoted $T_+ = J_{\frac{1}{2},-\frac{3}{2}}, U_+ = J_{\frac{1}{2},\frac{1}{2}}$.

We now consider the $SL(3, \mathbb{R})$ and $SU(2, 1)$ subalgebras of $G_{2(2)}$. The $SL(3, \mathbb{R})$ subalgebra is generated by the long roots in the non-compact basis. Its quadratic Casimir reads

$$C_2[SL(3, \mathbb{R})] = \frac{1}{4}H^2 + \frac{1}{3}Y_0^2 + \frac{1}{4}(E_{p^0}F_{p^0} + F_{p^0}E_{p^0} + E_{q^0}F_{q^0} + F_{q^0}E_{q^0}) + \frac{1}{2}(EF + FE) \quad (3.27)$$

The $SU(2, 1)$ subalgebra is generated by the long roots in the compact basis. Its quadratic Casimir reads

$$\begin{aligned} C_2[SU(2, 1)] = & -\frac{1}{3}S_3^2 - (J_3^2 + J_-J_+ + J_+J_-) \\ & + \frac{1}{8} \left(J_{\frac{1}{2},\frac{3}{2}}J_{-\frac{1}{2},-\frac{3}{2}} + J_{-\frac{1}{2},-\frac{3}{2}}J_{\frac{1}{2},\frac{3}{2}} \right) + \frac{1}{8} \left(J_{-\frac{1}{2},\frac{3}{2}}J_{\frac{1}{2},-\frac{3}{2}} + J_{\frac{1}{2},-\frac{3}{2}}J_{-\frac{1}{2},\frac{3}{2}} \right) \end{aligned} \quad (3.28)$$

3.2 Quaternionic symmetric space

The long $SU(2)_{J_{\pm}, J_3}$ endows $K \backslash G = (SU(2) \times SU(2)) \backslash G_{2(2)}$ with a quaternionic-Kähler geometry. In order to describe its geometry, we perform the Iwasawa decomposition $g = k \cdot e_{QK}$ (slightly adapted from [57])

$$e_{QK} = \tau_2^{-Y_0} \cdot e^{\sqrt{2}\tau_1 Y_+} \cdot e^{-UH} \cdot e^{-\zeta^0 E_{q_0} + \tilde{\zeta}_0 E_{p^0}} \cdot e^{-\sqrt{3}\zeta^1 E_{q_1} + \frac{\sqrt{3}}{3}\tilde{\zeta}_1 E_{p^1}} \cdot e^{\sigma E} \quad (3.29)$$

where k is an element of the maximal compact subgroup. This decomposition defines coordinates $(\tau_1, \tau_2, \zeta^0, \zeta^1, \tilde{\zeta}_0, \tilde{\zeta}_1, U, \sigma)$ on $K \backslash G$, where $\tau = \tau_1 + i\tau_2$ is an element of the upper half-plane.

The invariant \mathfrak{g} -valued one-form $\theta = de_{QK} \cdot e_{QK}^{-1}$ may be expanded on the compact basis, leading to the quaternionic viel-bein

$$\begin{pmatrix} \underline{J}_{-\frac{1}{2}, \frac{3}{2}} & \underline{J}_{\frac{1}{2}, \frac{3}{2}} \\ i\underline{J}_{-\frac{1}{2}, \frac{1}{2}} & \underline{J}_{\frac{1}{2}, \frac{1}{2}} \\ \underline{J}_{-\frac{1}{2}, -\frac{1}{2}} & i\underline{J}_{\frac{1}{2}, -\frac{1}{2}} \\ -\underline{J}_{-\frac{1}{2}, -\frac{3}{2}} & \underline{J}_{\frac{1}{2}, -\frac{3}{2}} \end{pmatrix} = - \begin{pmatrix} \bar{u} & v \\ -\bar{e}^1 & E_1 \\ \bar{E}_1 & e^1 \\ -\bar{v} & u \end{pmatrix}, \quad (3.30)$$

and the $SU(2) \times SU(2)$ spin connection

$$\begin{pmatrix} \underline{J}_{-} \\ \underline{J}_3 \\ \underline{J}_{+} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \bar{u} \\ \frac{1}{4i}(v - \bar{v}) + \frac{i\sqrt{3}}{4}(e^1 - \bar{e}^1) \\ u \end{pmatrix}, \quad \begin{pmatrix} \underline{S}_{-} \\ \underline{S}_3 \\ \underline{S}_{+} \end{pmatrix} = \frac{\sqrt{3}}{2} \begin{pmatrix} E_1 \\ \frac{i\sqrt{3}}{4}(v - \bar{v}) + \frac{i}{4}(e^1 - \bar{e}^1) \\ \bar{E}_1 \end{pmatrix}, \quad (3.31)$$

where

$$u = \frac{e^{-U}}{2\sqrt{2}\tau_2^{3/2}} \left(d\tilde{\zeta}_0 + \tau d\tilde{\zeta}_1 + 3\tau^2 d\zeta^1 - \tau^3 d\zeta^0 \right) \quad (3.32a)$$

$$v = dU + \frac{i}{2}e^{-2U} (d\sigma - \zeta^0 d\tilde{\zeta}_0 - \zeta^1 d\tilde{\zeta}_1 + \tilde{\zeta}_0 d\zeta^0 + \tilde{\zeta}_1 d\zeta^1) \quad (3.32b)$$

$$e^1 = \frac{i\sqrt{3}}{2\tau_2} d\tau \quad (3.32c)$$

$$E_1 = -\frac{e^{-U}}{2\sqrt{6}\tau_2^{3/2}} \left(3d\tilde{\zeta}_0 + d\tilde{\zeta}_1(\bar{\tau} + 2\tau) + 3\tau(2\bar{\tau} + \tau)d\zeta^1 - 3\bar{\tau}\tau^2 d\zeta^0 \right) \quad (3.32d)$$

This form of the vielbein agrees with that of the c -map space with prepotential $F = -(X^1)^3/X^0$ [45], as expected. The metric is then

$$ds^2 = 2(u\bar{u} + v\bar{v} + e^1\bar{e}^1 + E_1\bar{E}_1) \quad (3.33)$$

The right action of G on $K \backslash G$ is given by the vector fields

$$E = \partial_\sigma \quad (3.34a)$$

$$E_{p^0} = \partial_{\tilde{\zeta}_0} - \zeta^0 \partial_\sigma \quad , \quad E_{q_0} = -\partial_{\zeta^0} - \tilde{\zeta}_0 \partial_\sigma \quad (3.34b)$$

$$E_{p^1} = \sqrt{3}(\partial_{\tilde{\zeta}_1} - \zeta^1 \partial_\sigma) \quad , \quad E_{q_1} = \frac{1}{\sqrt{3}}(-\partial_{\zeta^1} - \tilde{\zeta}_1 \partial_\sigma) \quad (3.34c)$$

$$H = -\partial_U - 2\sigma \partial_\sigma - \zeta^0 \partial_{\zeta^0} - \zeta^1 \partial_{\zeta^1} - \tilde{\zeta}_0 \partial_{\tilde{\zeta}_0} - \tilde{\zeta}_1 \partial_{\tilde{\zeta}_1} \quad (3.34d)$$

$$Y_0 = -\frac{1}{2}(2\tau_1 \partial_{\tau_1} + 2\tau_2 \partial_{\tau_2} - 3\zeta^0 \partial_{\zeta^0} + 3\tilde{\zeta}_0 \partial_{\tilde{\zeta}_0} - \zeta^1 \partial_{\zeta^1} + \tilde{\zeta}_1 \partial_{\tilde{\zeta}_1}) \quad (3.34e)$$

$$Y_+ = \frac{1}{\sqrt{2}}(\partial_{\tau_1} + \zeta^0 \partial_{\zeta^1} - 6\zeta^1 \partial_{\tilde{\zeta}_1} - \tilde{\zeta}_1 \partial_{\tilde{\zeta}_0}) \quad (3.34f)$$

$$Y_- = \frac{1}{3\sqrt{2}}(6\tau_1 \tau_2 \partial_{\tau_2} + 3(\tau_1^2 - \tau_2^2) \partial_{\tau_1} + 9\tilde{\zeta}_0 \partial_{\tilde{\zeta}_1} - 9\zeta^1 \partial_{\zeta^0} + 2\tilde{\zeta}_1 \partial_{\zeta^1}) \quad (3.34g)$$

The other negative roots are too bulky to be displayed.

3.3 Twistor space

The twistor space

$$\mathcal{Z} = (SU(2)_{S_\pm, S_3} \times U(1)_{J_3}) \backslash G_{2(2)} \quad (3.35)$$

can be parameterized by the coset representative

$$e_{\mathcal{Z}} = e^{-\bar{z}J_+} (1 + z\bar{z})^{-iJ_3} e^{-zJ_-} e_{QK} \quad (3.36)$$

As in Section 2.3.1, we can construct complex coordinates on \mathcal{Z} using the Borel embedding:

$$C e_{\mathcal{Z}} C^{-1} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & * & * & 1 & & & \\ & * & * & & 1 & & \\ & * & * & & & 1 & \\ & * & * & * & * & * & 1 \\ & * & * & * & * & * & & 1 \end{pmatrix} \cdot \begin{pmatrix} * & * & & & & & \\ & * & * & & & & \\ & & * & * & * & & \\ & & & * & * & * & \\ & & & * & * & * & \\ & & & * & * & * & \\ & & & & * & * & \\ & & & & & * & * \end{pmatrix} \cdot \begin{pmatrix} 1 & * & * & * & * & * & \\ & 1 & * & * & * & * & \\ & & 1 & & * & * & \\ & & & 1 & * & * & \\ & & & & 1 & * & * \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \quad (3.37)$$

The entries of the upper triangular matrix then provide a complex coordinate system such that the “compact Heisenberg algebra” $J_{\frac{1}{2}, \pm \frac{1}{2} / \pm \frac{3}{2}}, J_+$ acts in a simple way, essentially by shifts. The Kähler potential in these coordinates (analogous to the bounded domain coordinates of $SL(2, \mathbb{R})/U(1)$) appears to be complicated; we do not consider them further here.

To get a simpler form for the Kähler potential we construct complex coordinates adapted to the “real” Heisenberg algebra, by performing a decomposition analogous to (3.37) but using $e_{\mathcal{Z}}$ directly rather than $Ce_{\mathcal{Z}}C^{-1}$, and then substituting $\bar{z} \rightarrow -z^{-1}$ at the end, as we did at the end of Section 2.3.4. The resulting complex coordinates $(\xi^0, \xi^1, \tilde{\xi}_0, \tilde{\xi}_1, \alpha)$ are related to the coordinates on the base and the stereographic coordinate on the fiber by

$$\begin{aligned}\xi^0 &= \zeta^0 + \frac{i}{2\sqrt{2}} \frac{e^U}{\tau_2^{3/2}} (z + z^{-1}) \quad , \quad \xi^1 = \zeta^1 + \frac{i}{2\sqrt{2}} \frac{e^U}{\tau_2^{3/2}} (\bar{\tau} z + \tau z^{-1}) \\ \tilde{\xi}_1 &= \tilde{\zeta}_1 - \frac{i}{2\sqrt{2}} \frac{e^U}{\tau_2^{3/2}} (3\bar{\tau}^2 z + 3\tau^2 z^{-1}) \quad , \quad \tilde{\xi}_0 = \tilde{\zeta}_0 + \frac{i}{2\sqrt{2}} \frac{e^U}{\tau_2^{3/2}} (\bar{\tau}^3 z + \tau^3 z^{-1}) \\ \alpha &= \sigma + \frac{i}{2\sqrt{2}} \frac{e^U}{\tau_2^{3/2}} \left[\left(\bar{\tau}^3 \zeta^0 - 3\bar{\tau}^2 \zeta^1 - \bar{\tau} \tilde{\zeta}_1 - \tilde{\zeta}_0 \right) z + \left(\tau^3 \zeta^0 - 3\tau^2 \zeta^1 - \tau \tilde{\zeta}_1 - \tilde{\zeta}_0 \right) z^{-1} \right]\end{aligned}\tag{3.38}$$

These coordinates are an example of the “canonical” coordinates of quasi-conformal geometries defined by Jordan algebras [25, 31]. Such coordinates in fact exist for all c -map spaces [36], and (3.38) agrees with the “twistor map” derived in that context in [36]. To compare the two, recall that the Kähler potential on the special Kähler base (the upper half-plane in this case) is $e^{-K} = 8\tau_2^3$.

The Kähler potential on \mathcal{Z} is

$$K_{\mathcal{Z}} = \frac{1}{2} \log N_4 = \frac{1}{2} \log \left[I_4(\xi^I - \bar{\xi}^I, \tilde{\xi}_I - \bar{\tilde{\xi}}_I) + (\alpha - \bar{\alpha} + \xi^I \bar{\tilde{\xi}}_I - \bar{\xi}^I \tilde{\xi}_I)^2 \right]\tag{3.39}$$

where

$$I_4(\xi, \tilde{\xi}) = (\xi^0)^2 \tilde{\xi}_0^2 + 4(\xi^1)^3 \tilde{\xi}_0 + 2\xi^0 \tilde{\xi}_0 \xi^1 \tilde{\xi}_1 - \frac{1}{3}(\xi^1)^2 \tilde{\xi}_1^2 - \frac{4}{27} \tilde{\xi}_1^3 \xi^0\tag{3.40}$$

is the quartic invariant of $SL(2, \mathbb{R})$ in its spin-3/2 representation, and N_4 is the quartic distance function of quasi-conformal geometry, which defines the “quartic light-cone”.

In parallel to the discussion of Section 2.3.5, one can also define the homogeneous line bundles $\mathcal{O}(k)$ over \mathcal{Z} , and the total space of $\mathcal{O}(-2)$ gives the Swann space \mathcal{S} .

3.4 Quasi-conformal representations

We construct a principal series representation of G by induction from the parabolic P , with the character $\chi_k = e^{-k\frac{H}{2}}$ of P . The infinitesimal action of G in this representation

is determined as in Section 2.4.1, this time using the decomposition of G as

$$g = p \cdot e^{-\zeta^0 E_{q_0} + \tilde{\zeta}_0 E_{p^0}} \cdot e^{-\sqrt{3}\zeta^1 E_{q_1} + \frac{\sqrt{3}}{3}\tilde{\zeta}_1 E_{p^1}} \cdot e^{\sigma E} \quad (3.41)$$

$$= p \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\sqrt{2}}{3}\tilde{\zeta}_1 & 0 & \sqrt{2}\tilde{\zeta}_0 \\ 0 & 1 & 0 & 0 & -2\zeta^1 & 0 & -\frac{2}{3}\tilde{\zeta}_1 \\ 0 & 0 & 1 & 0 & -\sqrt{2}\zeta^0 & 0 & \sqrt{2}\zeta^1 \\ \sqrt{2}\zeta^1 & \frac{2}{3}\tilde{\zeta}_1 & \sqrt{2}\tilde{\zeta}_0 & 1 & \sigma - \zeta^0\tilde{\zeta}_0 - \frac{1}{3}\zeta^1\tilde{\zeta}_1 & 0 & 2\tilde{\zeta}_0\zeta^1 - \frac{2}{9}\tilde{\zeta}_1^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \sqrt{2}\zeta^0 & -2\zeta^1 & -\frac{\sqrt{2}}{3}\tilde{\zeta}_1 & 0 & 2(\zeta^1)^2 + \frac{2}{3}\zeta^0\tilde{\zeta}_1 & 1 & \sigma + \zeta^0\tilde{\zeta}_0 + \frac{1}{3}\zeta^1\tilde{\zeta}_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.42)$$

where $p \in P$. The generators act by

$$\begin{aligned} H^{QC} &= -2\sigma\partial_\sigma - \zeta^0\partial_{\zeta^0} - \zeta^1\partial_{\zeta^1} - \tilde{\zeta}_0\partial_{\tilde{\zeta}_0} - \tilde{\zeta}_1\partial_{\tilde{\zeta}_1} - k, & E^{QC} &= -\partial_\sigma \\ E_{p^0}^{QC} &= \partial_{\tilde{\zeta}_0} + \zeta^0\partial_\sigma, & E_{p^1}^{QC} &= \sqrt{3}(\partial_{\tilde{\zeta}_1} + \zeta^1\partial_\sigma) \\ E_{q_0}^{QC} &= -\partial_{\zeta^0} + \tilde{\zeta}_0\partial_\sigma, & E_{q_1}^{QC} &= \frac{1}{\sqrt{3}}(-\partial_{\zeta^1} + \tilde{\zeta}_1\partial_\sigma) \\ Y_+^{QC} &= \frac{1}{\sqrt{2}}\zeta^0\partial_{\zeta^1} - 3\sqrt{2}\zeta^1\partial_{\tilde{\zeta}_1} - \frac{1}{\sqrt{2}}\tilde{\zeta}_1\partial_{\tilde{\zeta}_0}, & Y_-^{QC} &= \frac{3}{\sqrt{2}}(\tilde{\zeta}_0\partial_{\tilde{\zeta}_1} - \zeta^1\partial_{\zeta^0}) + \frac{\sqrt{2}}{3}\tilde{\zeta}_1\partial_{\zeta^1} \\ Y_0^{QC} &= \frac{1}{2}(3\zeta^0\partial_{\zeta^0} + \zeta^1\partial_{\zeta^1} - 3\tilde{\zeta}_0\partial_{\tilde{\zeta}_0} - \tilde{\zeta}_1\partial_{\tilde{\zeta}_1}) \\ F^{QC} &= \left(2(\zeta^1)^3 + \zeta^0\tilde{\zeta}_1\zeta^1 + (\zeta^0)^2\tilde{\zeta}_0 + \sigma\zeta^0\right)\partial_{\zeta^0} - \left(\frac{1}{3}\tilde{\zeta}_1(\zeta^1)^2 - \zeta^0\tilde{\zeta}_0\zeta^1 + \frac{2}{9}\zeta^0\tilde{\zeta}_1^2 - \sigma\zeta^1\right)\partial_{\zeta^1} \\ &\quad + \left(-6\tilde{\zeta}_0(\zeta^1)^2 + \frac{1}{3}\tilde{\zeta}_1^2\zeta^1 - \zeta^0\tilde{\zeta}_0\tilde{\zeta}_1 + \sigma\tilde{\zeta}_1\right)\partial_{\tilde{\zeta}_1} + \left(\frac{2}{27}\tilde{\zeta}_1^3 - \tilde{\zeta}_0\zeta^1\tilde{\zeta}_1 - \zeta^0\tilde{\zeta}_0^2 + \sigma\tilde{\zeta}_0\right)\partial_{\tilde{\zeta}_0} \\ &\quad - [I_4(\zeta, \tilde{\zeta}) + \sigma^2]\partial_\sigma - k\sigma \end{aligned} \quad (3.43)$$

where $I_4(\zeta, \tilde{\zeta})$ is the quartic polynomial defined in (3.40). The quadratic Casimir evaluates to

$$C_2 = -\frac{1}{4}k(6-k) \quad (3.44)$$

This representation is unitary for $k \in 3 + i\mathbb{R}$, with respect to the inner product

$$\langle f_1 | f_2 \rangle = \int d\zeta^\Lambda d\tilde{\zeta}_\Lambda d\sigma f_1^*(\zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) f_2(\zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma). \quad (3.45)$$

Complexifying, we can also consider the holomorphic right action of G on sections of the line bundle $\mathcal{O}(-k)$ on the twistor space $\mathcal{Z} = P_{\mathbb{C}} \backslash G_{\mathbb{C}}$. The infinitesimal action is

the complexification of the one above,

$$\begin{aligned}
H^{QC} &= -2\alpha\partial_\alpha - \xi^0\partial_{\xi^0} - \xi^1\partial_{\xi^1} - \tilde{\xi}_0\partial_{\tilde{\xi}_0} - \tilde{\xi}_1\partial_{\tilde{\xi}_1} - k, & E^{QC} &= -\partial_\alpha \\
E_{p^0}^{QC} &= \partial_{\tilde{\xi}_0} + \xi^0\partial_\alpha, & E_{p^1}^{QC} &= \sqrt{3}(\partial_{\tilde{\xi}_1} + \xi^1\partial_\alpha) \\
E_{q_0}^{QC} &= -\partial_{\xi^0} + \tilde{\xi}_0\partial_\alpha, & E_{q_1}^{QC} &= \frac{1}{\sqrt{3}}(-\partial_{\xi^1} + \tilde{\xi}_1\partial_\alpha) \\
Y_+^{QC} &= \frac{1}{\sqrt{2}}\xi^0\partial_{\xi^1} - 3\sqrt{2}\xi^1\partial_{\tilde{\xi}_1} - \frac{1}{\sqrt{2}}\tilde{\xi}_1\partial_{\tilde{\xi}_0}, & Y_-^{QC} &= \frac{3}{\sqrt{2}}(\tilde{\xi}_0\partial_{\tilde{\xi}_1} - \xi^1\partial_{\xi^0}) + \frac{\sqrt{2}}{3}\tilde{\xi}_1\partial_{\xi^1} \\
Y_0^{QC} &= \frac{1}{2}(3\xi^0\partial_{\xi^0} + \xi^1\partial_{\xi^1} - 3\tilde{\xi}_0\partial_{\tilde{\xi}_0} - \tilde{\xi}_1\partial_{\tilde{\xi}_1}) \\
F^{QC} &= \left(2(\xi^1)^3 + \xi^0\tilde{\xi}_1\xi^1 + (\xi^0)^2\tilde{\xi}_0 + \alpha\xi^0\right)\partial_{\xi^0} - \left(\frac{1}{3}\tilde{\xi}_1(\xi^1)^2 - \xi^0\tilde{\xi}_0\xi^1 + \frac{2}{9}\xi^0\tilde{\xi}_1^2 - \alpha\xi^1\right)\partial_{\xi^1} \\
&\quad + \left(-6\tilde{\xi}_0(\xi^1)^2 + \frac{1}{3}\tilde{\xi}_1^2\xi^1 - \xi^0\tilde{\xi}_0\tilde{\xi}_1 + \alpha\tilde{\xi}_1\right)\partial_{\tilde{\xi}_1} + \left(\frac{2}{27}\tilde{\xi}_1^3 - \tilde{\xi}_0\xi^1\tilde{\xi}_1 - \xi^0\tilde{\xi}_0^2 + \alpha\tilde{\xi}_0\right)\partial_{\tilde{\xi}_0} \\
&\quad - [I_4(\xi, \tilde{\xi}) + \alpha^2]\partial_\alpha - k\alpha
\end{aligned} \tag{3.46}$$

Naively this construction would require $k \in \mathbb{Z}$, but in what follows, we will sometimes consider this action not only when k is integral but even when $k \in \frac{1}{3}\mathbb{Z}$. Presumably this should be understood in terms of a triple cover of \mathcal{Z} .

This representation is formally unitary under the inner product

$$\langle f_1 | f_2 \rangle = \int d\xi^\Lambda d\tilde{\xi}_\Lambda d\alpha d\bar{\xi}^\Lambda d\bar{\xi}_\Lambda d\bar{\alpha} e^{(k-6)Kz} f_1^*(\bar{\xi}, \bar{\xi}, \bar{\alpha}) f_2(\xi, \tilde{\xi}, \alpha) \tag{3.47}$$

As discussed in Section 2.4.2 for $G = SU(2, 1)$, we expect this formally unitary action on sections of $\mathcal{O}(-k)$ to yield a genuine unitary action on $H^1(\mathcal{Z}, \mathcal{O}(-k))$ at least for $k \geq 3$; for $k \geq 5$ it should give the quaternionic discrete series of $G_{2(2)}$. Moreover, it should occur as a subquotient of the principal series for some k . We will discuss this further at the level of the K -finite vectors below.

3.4.1 Lift to hyperkahler cone

Similar to the discussion of Section 2.6.1 for $G = SU(2, 1)$, the action of $G = G_{2(2)}$ on holomorphic sections of $\mathcal{O}(-k)$ over \mathcal{Z} is equivalent to an action on holomorphic functions of homogeneity degree $-k$ on the Swann space \mathcal{S} . Introducing the complex coordinates on \mathcal{S}

$$v^\flat = e^{2t}, \quad w_\flat = -\frac{1}{4i}(\alpha + \xi^0\tilde{\xi}_0 + \xi^1\tilde{\xi}_1) \tag{3.48}$$

$$v^0 = 3\sqrt{3}\xi^0 e^{2t}, \quad w_0 = -\frac{i}{6\sqrt{3}}\tilde{\xi}_0, \quad v^1 = -\frac{1}{\sqrt{3}}\xi^1 e^{2t}, \quad w_1 = \frac{i\sqrt{3}}{2}\tilde{\xi}_1, \tag{3.49}$$

it is straightforward to compute the holomorphic vector fields corresponding to the action of $G_{\mathbb{C}}$ on \mathcal{S} and determine their holomorphic moment maps. Expressing the result in terms of the coordinates $\xi^I, \tilde{\xi}_I, \alpha$ on \mathcal{Z} and t in the \mathbb{C}^* fiber, we find the holomorphic moment maps

$$\begin{aligned}
\mathcal{H} &= \frac{i}{2}e^{2t}\alpha, \quad \mathcal{Y}_0 = \frac{i}{4}e^{2t}(3\xi^0\tilde{\xi}_0 + \xi^1\tilde{\xi}_1), \quad \mathcal{E} = -\frac{i}{4}e^{2t}, \\
\mathcal{Y}_+ &= -\frac{i}{2\sqrt{2}}e^{2t}(\tilde{\xi}_1^2 + 9\tilde{\xi}_0\xi^1), \quad \mathcal{Y}_- = \frac{i}{6\sqrt{2}}e^{2t}(\xi^0\tilde{\xi}_1 - 3(\xi^1)^2) \\
\mathcal{E}_{p^0} &= -\frac{3i\sqrt{3}}{2}e^{2t}\tilde{\xi}_0, \quad \mathcal{E}_{p^1} = \frac{i}{2}e^{2t}\tilde{\xi}_1, \quad \mathcal{E}_{q_0} = \frac{i}{6\sqrt{3}}e^{2t}\xi^0, \quad \mathcal{E}_{q_1} = -\frac{i}{2}e^{2t}\xi^1 \\
\mathcal{F}_{p^0} &= \frac{i}{6\sqrt{3}}e^{2t}(-2(\xi^1)^3 + \xi^0\tilde{\xi}_1\xi^1 + \alpha\xi^0 + (\xi^0)^2\tilde{\xi}_0) \\
\mathcal{F}_{p^1} &= \frac{i}{18}e^{2t}(3\tilde{\xi}_1(\xi^1)^2 - 9\alpha\xi^1 - 9\xi^0\tilde{\xi}_0\xi^1 - 2\xi^0\tilde{\xi}_1^2) \\
\mathcal{F}_{q_0} &= -\frac{i}{6\sqrt{3}}e^{2t}(2\tilde{\xi}_1^3 + 27\tilde{\xi}_0\xi^1\tilde{\xi}_1 + 27\xi^0\tilde{\xi}_0^2 - 27\alpha\tilde{\xi}_0) \\
\mathcal{F}_{q_1} &= \frac{i}{2}e^{2t}\left[\tilde{\xi}_0(\xi^0\tilde{\xi}_1 - 6(\xi^1)^2) - \frac{1}{3}\tilde{\xi}_1(3\alpha + \xi^1\tilde{\xi}_1)\right] \\
\mathcal{F} &= \frac{i}{2}e^{2t}\left[2\tilde{\xi}_0\xi^{13} + \frac{\tilde{\xi}_1^2(\xi^1)^2}{6} - \xi^0\tilde{\xi}_0\tilde{\xi}_1\xi^1 - \frac{2\xi^0\tilde{\xi}_1^3}{27} + \frac{\alpha^2}{2} - \frac{(\xi^0)^2\tilde{\xi}_0^2}{2}\right] \tag{3.50}
\end{aligned}$$

In order to compute the moment maps in the compact basis, which will be relevant in the next section, it is convenient to change variables to (a, b, \bar{a}, \bar{b}) transforming homogeneously under the compact generator R_0 :

$$\xi^0 = -\frac{1}{8}(a + \bar{a} - b - \bar{b}), \quad \xi^1 = \frac{1}{24}i(a - \bar{a} + 3b - 3\bar{b}) \tag{3.51}$$

$$\tilde{\xi}_1 = \frac{1}{8}(a + \bar{a} + 3(b + \bar{b})), \quad \tilde{\xi}_0 = \frac{1}{8}i(a - \bar{a} - b + \bar{b}) \tag{3.52}$$

$$R_0 = \frac{i}{2}(a\partial_a - \bar{a}\partial_{\bar{a}} - 3b\partial_b + 3\bar{b}\partial_{\bar{b}}) \tag{3.53}$$

The holomorphic moment maps for the generators in the compact basis then read

$$\begin{aligned}
\mathcal{J}_3 &= \frac{ie^{2t}}{27648}[1728(1 + \alpha^2) - a^2\bar{a}^2 - 4(a^3b + \bar{a}^3\bar{b}) - 1296b\bar{b} + 27b^2\bar{b}^2 - 18a\bar{a}(b\bar{b} - 8)] \\
\mathcal{S}_3 &= \frac{ie^{2t}}{9216}[1728(1 + \alpha^2) - a^2\bar{a}^2 - 4(a^3b + \bar{a}^3\bar{b}) + 432b\bar{b} + 27b^2\bar{b}^2 - 6a\bar{a}(3b\bar{b} + 8)] \\
\mathcal{J}_+ &= -\frac{i}{1728\sqrt{2}}e^{2t}[2a^3 + 9a\bar{a}\bar{b} + 27\bar{b}(8 + 8i\alpha - b\bar{b})], \\
\mathcal{S}_+ &= \frac{i}{576\sqrt{2}}e^{2t}[a\bar{a}^2 + 6a^2b + 9\bar{a}(8 + 8i\alpha + b\bar{b})],
\end{aligned}$$

$$\begin{aligned}
J_{\frac{1}{2}, \frac{3}{2}} &= -\frac{e^{2t}}{6912} [1728(1 - \alpha^2) + 3456i\alpha + a^2\bar{a}^2 + 4a^3b + 4\bar{a}^3\bar{b} + 18a\bar{a}b\bar{b} - 27b^2\bar{b}^2] , \\
\mathcal{J}_{\frac{1}{2}, \frac{1}{2}} &= \frac{ie^{2t}}{288\sqrt{6}} (a^2\bar{a} + 6\bar{a}^2\bar{b} + 9a(-8 - 8i\alpha + b\bar{b})) , \quad \mathcal{J}_{\frac{1}{2}, -\frac{1}{2}} = \frac{i}{24\sqrt{3}} e^{2t} (a^2 + 3\bar{a}\bar{b}) , \\
\mathcal{J}_{\frac{1}{2}, -\frac{3}{2}} &= \frac{ie^{2t}}{864\sqrt{2}} (2a^3 + 9a\bar{a}\bar{b} - 27\bar{b}(8 - 8i\alpha + b\bar{b})) .
\end{aligned} \tag{3.54}$$

Substituting (3.50) in (3.1), a tedious computation shows that the holomorphic moment map, seen as an element of $\mathfrak{g}_{\mathbb{C}}$, is nilpotent of degree 2. Thus, the Swann space \mathcal{S} is isomorphic (at least locally) to the complexified minimal nilpotent orbit of $G_{2(2)}$. In particular, we note the classical identities

$$\underline{E}_{p^1}^2 + \sqrt{3}\underline{E}_{p^0}\underline{E}_{q_1} - 2\sqrt{2}\underline{E}\underline{Y}_+ = 0 \tag{3.55a}$$

$$\underline{E}_{q_1}^2 - \sqrt{3}\underline{E}_{q_0}\underline{E}_{p^1} - 2\sqrt{2}\underline{E}\underline{Y}_- = 0 \tag{3.55b}$$

$$3\underline{E}_{p^0}\underline{E}_{q_0} + \underline{E}_{p^1}\underline{E}_{q_1} - 4\underline{E}\underline{Y}_0 = 0 \tag{3.55c}$$

$$9C_2(\underline{L}) - C_2(\underline{R}) = 9C_2(\underline{J}) - C_2(\underline{S}) = 0 . \tag{3.55d}$$

3.4.2 Some finite K -types

We now discuss the finite K -types of the principal series representation. The full K -type decomposition can be obtained using Frobenius reciprocity¹⁹; here we are interested in the explicit construction of some specific states.

As in Section 2.4.4, we can construct the spherical vector f_K by noting that the Heisenberg parabolic subgroup P acting on the two-form $e_4 \wedge e_6$ gives a 1-dimensional representation, where e_i is the i -th row of the matrix in (3.1), and K preserves the norm of each row; so we take

$$f_K = \|e_4 \wedge e_6\|^{-k/2} \tag{3.56}$$

where

$$\|e_4 \wedge e_6\|^2 = \tilde{I}_4 + (I_6 + \alpha\hat{I}_4) - 2[1 + I_2 + (\alpha^2 - I_4)]^2 \tag{3.57}$$

Here, I_4 is the quartic polynomial (3.40), invariant under $SL(2, \mathbb{R})$, while $I_2, \tilde{I}_4, \hat{I}_4, I_6$ are homogeneous polynomials of respective degree 2, 4, 4, 6 in $\xi^I, \tilde{\xi}_I$, invariant under the

¹⁹The Frobenius reciprocity theorem implies that the number of occurrences of a K -type σ in the principal series we consider is equal to the number of singlets in the decomposition of σ under $K \cap M$, where M is the centralizer in K of A , appearing in the Langlands decomposition $P = MAN$; see e.g. [49].

maximal compact subgroup $SO(2) \subset SL(2, \mathbb{R})$. In terms of the variables (3.51)

$$I_2 = \frac{1}{12}(a\bar{a} + 3b\bar{b}) , \quad \tilde{I}_4 = -\frac{1}{27}(ba^3 + \bar{a}^3\bar{b}) , \quad \hat{I}_4 = \frac{i}{27}(a^3b - \bar{a}^3\bar{b}) \quad (3.58a)$$

$$I_4 = \frac{1}{1728} [4(ba^3 + \bar{b}\bar{a}^3) + \bar{a}^2a^2 - 27\bar{b}^2b^2 + 18a\bar{a}b\bar{b}] \quad (3.58b)$$

$$I_6 = -\frac{1}{5832} [2\bar{a}^3a^3 + 54\bar{a}^2a^2b\bar{b} + 9(a\bar{a} + 3b\bar{b})(ba^3 + \bar{b}\bar{a}^3)] \quad (3.58c)$$

Using (3.54), one easily recognizes that the spherical vector is related to the quadratic Casimirs of $SU(2)_J$ and $SU(2)_S$ simply by

$$e^{-kt} f_K = [C_2(\mathcal{J})]^{-k/4} = [C_2(\mathcal{S})/9]^{-k/4} \quad (3.59)$$

which makes the K invariance manifest.

Other K -types may be obtained by acting on f_K with the non-compact generators $J_{\pm\frac{1}{2}, \pm\frac{3}{2}}$. For example, a set of J_+ -highest weights are obtained by acting with symmetrized products of the raising operators $J_{\frac{1}{2}, \pm\frac{3}{2}}, J_{\frac{1}{2}, \pm\frac{1}{2}}$, which transform in the spin-3/2 representation of $SU(2)_S$ (antisymmetric combinations of these operators lead instead to J_+ -descendants); these generate the spectrum

$$\bigoplus_n \left[\frac{n}{2} \right]_J \otimes S^n \left(\left[\frac{3}{2} \right]_S \right). \quad (3.60)$$

To see this spectrum appearing more explicitly, note that a class of (J_+, S_+) -highest weight states can be obtained by acting on f_K with

$$M_1 = J_{\frac{1}{2}, \frac{3}{2}} \quad (3.61a)$$

$$M_2 = J_{\frac{1}{2}, \frac{1}{2}}^2 + i\sqrt{3}J_{\frac{1}{2}, -\frac{1}{2}}J_{\frac{1}{2}, \frac{3}{2}} \quad (3.61b)$$

$$M_3 = J_{\frac{1}{2}, \frac{3}{2}}^2 J_{\frac{1}{2}, -\frac{3}{2}} + iJ_{\frac{1}{2}, \frac{3}{2}}J_{\frac{1}{2}, \frac{1}{2}}J_{\frac{1}{2}, -\frac{1}{2}} + \frac{2}{3\sqrt{3}}J_{\frac{1}{2}, \frac{1}{2}}^3 \quad (3.61c)$$

$$\begin{aligned} M_4 = & J_{\frac{1}{2}, \frac{3}{2}}^2 J_{\frac{1}{2}, -\frac{3}{2}}^2 + \frac{1}{3}J_{\frac{1}{2}, \frac{1}{2}}^2 J_{\frac{1}{2}, -\frac{1}{2}}^2 + 2iJ_{\frac{1}{2}, \frac{3}{2}}J_{\frac{1}{2}, \frac{1}{2}}J_{\frac{1}{2}, -\frac{1}{2}}J_{\frac{1}{2}, -\frac{3}{2}} \\ & + \frac{4i}{3\sqrt{3}}J_{\frac{1}{2}, \frac{3}{2}}J_{\frac{1}{2}, -\frac{1}{2}}^3 + \frac{4}{3\sqrt{3}}J_{\frac{1}{2}, \frac{1}{2}}^3J_{\frac{1}{2}, -\frac{3}{2}} \end{aligned} \quad (3.61d)$$

which satisfy

$$[M_i, M_j] \equiv 0 , \quad M_1^2 M_4 + \frac{4}{27}M_2^3 - M_3^2 \equiv 0 \quad (3.62)$$

when acting²⁰ on states annihilated by J_+ . A generating function for the (S, J) spectrum obtained by acting with the M_i on f_K is

$$\text{Tr } z^S q^{2J} = \frac{1 - q^6 z^3}{(1 - q z^{3/2})(1 - q^2 z)(1 - q^3 z^{3/2})(1 - q^4)} \quad (3.63)$$

where the denominator corresponds to the action of M_i while the numerator reflects the constraint (3.62). This generating function indeed agrees with (3.60).²¹

As an example, the action of M_i on f_K may be easily computed,

$$M_1 f_K = k \left(\frac{\mathcal{S}_3 \mathcal{J}_{\frac{1}{2}, \frac{3}{2}} - \frac{i}{\sqrt{3}} \mathcal{J}_{\frac{1}{2}, \frac{1}{2}} \mathcal{S}_+}{C_2(\mathcal{J})} \right) f_K \quad (3.64a)$$

$$M_2 f_K = k(3k - 2) \left(\frac{\mathcal{J}_+ \mathcal{S}_+}{C_2(\mathcal{J})} \right) f_K \quad (3.64b)$$

$$M_3 f_K = k(9k^2 - 4) \left(\frac{\mathcal{J}_+ \mathcal{J}_{\frac{1}{2}, \frac{3}{2}}}{C_2(\mathcal{J})} \right) f_K \quad (3.64c)$$

$$M_4 f_K = k^2(9k^2 - 4) \left(\frac{\mathcal{J}_+^2}{C_2(\mathcal{J})} \right) f_K \quad (3.64d)$$

where equalities hold up to unimportant numerical factors.

Acting with the M_i on f_K is not sufficient to construct all of the (J_+, S_+) highest weight vectors. To see this explicitly, it is enough to note that there are other operators built from $J_{\pm \frac{1}{2}, \frac{3}{2}}, J_{\pm \frac{1}{2}, \frac{1}{2}}$ which commute with S_+ and J_+ , for example

$$P_2 = J_{\frac{1}{2}, \frac{3}{2}} J_{-\frac{1}{2}, \frac{1}{2}} - i J_{-\frac{1}{2}, \frac{3}{2}} J_{\frac{1}{2}, \frac{1}{2}} \quad (3.65)$$

which acts on f_K by

$$P_2 f_K = k(k - 2) \left(\frac{\mathcal{S}_+^2}{C_2(\mathcal{J})} \right) f_K \quad (3.66)$$

Nevertheless, we focus on (J_+, S_+) -highest weight states generated by the action of M_i only, of the form

$$f_{p_1, p_2, p_3, p_4} = \left(\mathcal{S}_3 \mathcal{J}_{\frac{1}{2}, \frac{3}{2}} - \frac{i}{\sqrt{3}} \mathcal{J}_{\frac{1}{2}, \frac{1}{2}} \mathcal{S}_+ \right)^{p_1} \mathcal{J}_{\frac{1}{2}, \frac{3}{2}}^{p_2} \mathcal{J}_+^{p_3} \mathcal{S}_+^{p_4} [C_2(\mathcal{J})]^{-\frac{k}{4} - p_1 - \frac{1}{2}(p_2 + p_3 + p_4)} \quad (3.67)$$

²⁰For example, the second equation of (3.62) follows from $M_1^2 M_4 + \frac{4}{27} M_2^3 - M_3^2 = \frac{416}{9} J_{\frac{1}{2}, \frac{3}{2}}^2 J_+^2 - 8 J_{\frac{1}{2}, \frac{3}{2}}^3 J_{\frac{1}{2}, -\frac{3}{2}} J_+ - \frac{56i}{3} J_{\frac{1}{2}, \frac{3}{2}}^2 J_{\frac{1}{2}, \frac{1}{2}} J_{\frac{1}{2}, -\frac{1}{2}} J_+ - \frac{40\sqrt{3}}{9} J_{\frac{1}{2}, \frac{3}{2}}^3 J_{\frac{1}{2}, \frac{1}{2}} J_+$.

²¹A simple check is obtained by manipulating (3.63) to get $\sum_{n, S} (2S + 1) q^{2J} = 1/(1 - q)^4$, the partition function of four free bosons.

where p_i are integers related to the $SU(2)_J \times SU(2)_S$ spin by

$$j = \frac{1}{2}(p_1 + p_2 + 2p_3) , \quad s = \frac{1}{2}(3p_1 + 3p_2 + 4p_4) . \quad (3.68)$$

The set of such states is pictured in Figure 5.

Now we recall some results of [26]. First, the K -type decomposition of $H^1(\mathcal{Z}, \mathcal{O}(-k))$ for $k \geq 3$ is given by

$$\bigoplus_{n=0}^{\infty} \left[\frac{k-2+n}{2} \right]_J \otimes S^n \left(\left[\frac{3}{2} \right]_S \right) . \quad (3.69)$$

For $k \geq 5$ this gives a quaternionic discrete representation of G , which we expect to find as a submodule of the principal series. Indeed, when $k-2$ is a multiple of 4, these K -types do appear in Figure 5: they are the ones reached by acting with the M_i on f_K and in particular using M_4 at least $\frac{1}{4}(k-2)$ times.

The paper [26] also describes a pattern of other representations which would be expected to appear as submodules of the principal series for general quaternionic groups. While that analysis does not directly apply to $G = G_{2(2)}$ we describe its naive extrapolation here, and explain how the expected K -type decompositions can be naturally obtained by acting with the operators M_i on an appropriate lowest K -type.

The representations in question are supposed to correspond to the orbits of $SL(2, \mathbb{C})$ acting on the complexified spin- $\frac{3}{2}$ representation (\mathbb{C}^4) . Their K -type decompositions would be of the form

$$\bigoplus_{n=0}^{\infty} \left[\frac{k-2+n}{2} \right]_J \otimes A_n(X) \quad (3.70)$$

where $A(X) = \sum_{n=0}^{\infty} A_n(X)$ is the algebra of functions on an orbit X of the action of $SL(2, \mathbb{C})$ on \mathbb{C}^4 , considered as a representation of $SU(2)_S$. There are three examples:

- (i) A representation π'_1 at $k = 1$, corresponding to the orbit X defined by $I_4 = 0$. The K -type decomposition is obtained by removing the contribution of the operator M_4 from (3.63), and acting on a highest weight vector with $(J, S) = (0, \frac{1}{2})$ instead of f_K : this gives

$$\mathrm{Tr}_{\pi'_1} z^S q^{2J} = z^{\frac{1}{2}} \frac{1 - q^6 z^3}{(1 - q z^{3/2})(1 - q^2 z)(1 - q^3 z^{3/2})} \quad (3.71)$$

This decomposition is multiplicity-free. π'_1 has Gelfand-Kirillov dimension 4, and K -types contained in a wedge — see the black dots on Figure 5 (shifted by $\Delta S = \frac{1}{2}$). It appears on the list of unitary representations of $G_{2(2)}$ in [8].

- (ii) A representation $\pi'_{2/3}$ at $k = 2/3$, corresponding to the locus where $I_4 = dI_4 = 0$.²² Its K -type decomposition can be obtained by acting on a highest weight vector with $(J, S) = (0, 2)$ with the operators M_2 and M_1 , but imposing the requirement that $M_2^2 = 0$. It is represented in Figure 5 by the two leftmost “Regge trajectories” of slope 3 (shifted by $\Delta S = 2$).
- (iii) A representation $\pi'_{4/3}$ at $k = 4/3$, corresponding to the locus where $I_4 = dI_4 = d^2I_4 = 0$. This is the minimal or “ladder” representation of $G_{2(2)}$, with K -type decomposition

$$\bigoplus_{m=0}^{\infty} \left[\frac{m}{2} \right]_J \otimes \left[\frac{3m+2}{2} \right]_S \quad (3.72)$$

The highest weight states can be obtained by acting on the highest weight of a K -type with $J = 0$, $S = 1$ by the operator M_1 only; it is represented in Figure 5 by the leading “Regge trajectory” of slope 3 (shifted by $\Delta S = 1$).

A direct approach to the construction of the submodules π'_k inside the principal series will be discussed in Section 3.6.

²²The correspondence between orbits and representations in this case is somewhat degenerate, because actually this orbit is the same as the minimal orbit we discuss next (unlike what happens for other G , where the orbit defined by $dI_4 = 0$ is different from that defined by $d^2I_4 = 0$).

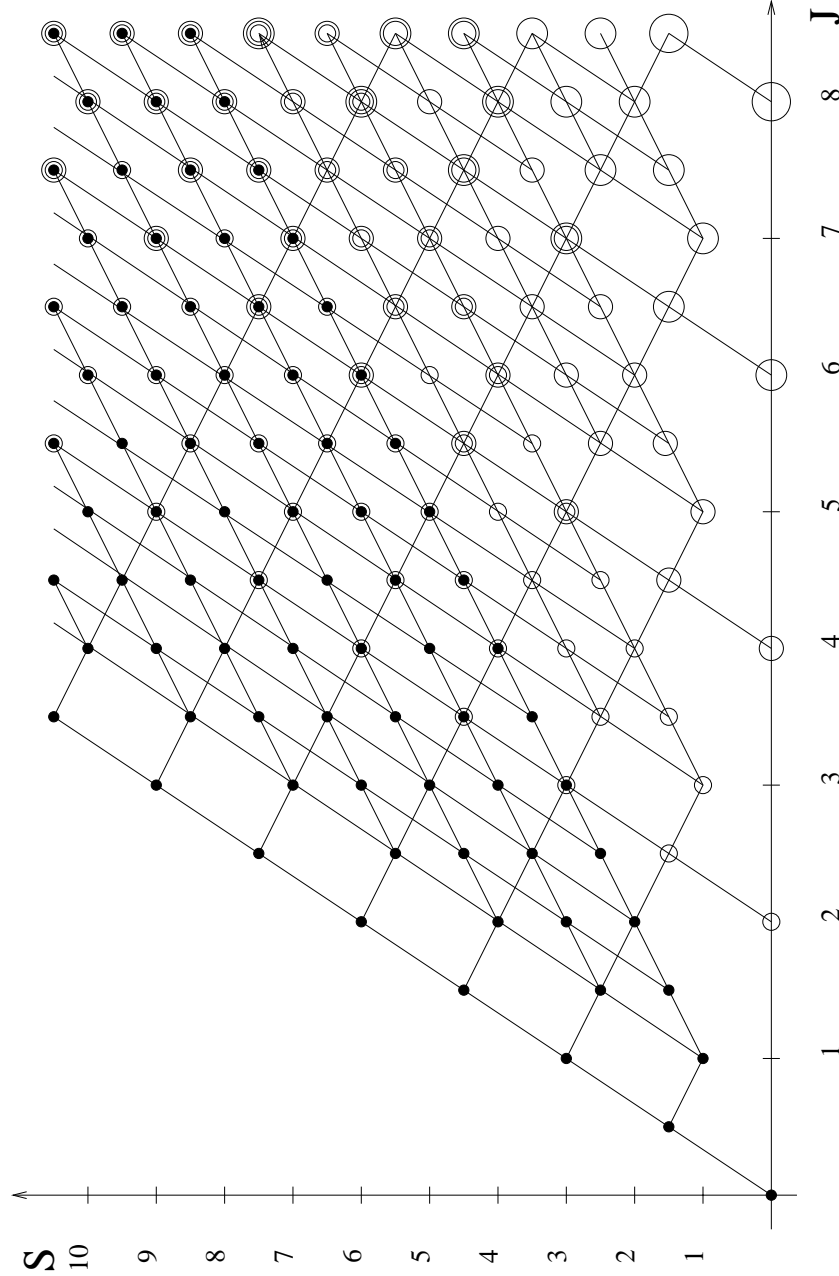


Figure 5: The states (3.67) obtained by acting with the M_i on f_K . Multiplicities are indicated by the number of concentric circles. The radius of the circle indicates the number of powers of M_4 that need to be applied to f_K in order to reach the state. This figure also represents the set of highest weight vectors one could obtain by acting with the M_i on some other ground state; in this case the labels J and S are shifted by the quantum numbers of the ground state, and in some cases one gets only a subset of the states pictured. In particular, the modules $\pi'_1, \pi'_{2/3}, \pi'_{4/3}$ discussed in the text correspond to the wedge spanned by the black dots, to the first and second “Regge trajectories” of slope 3, and to the first “Regge trajectory” respectively.

3.5 Minimal representation

The minimal representation of $G_{2(2)}$, of functional dimension 3, was first constructed in [55], and further analyzed in [26]. As we just recalled, it can be obtained as a submodule of a degenerate principal series representation [26]. According to the orbit philosophy, it arises by quantizing the minimal nilpotent orbit of $G_{2(2)}$, or equivalently by holomorphic quantization of the Swann space \mathcal{S} . We start by recalling two different realizations of the minimal representation by differential operators, the first one acting on functions $f(y, x_0, x_1)$ of real variables, the second on functions $f(x, a, b)$ of complex variables.

3.5.1 Real polarization

In the real polarization used in [39],

$$\begin{aligned}
E &= -\frac{iy}{2}, \quad Y_0 = -\frac{1}{2}(x_1\partial_1 + 3x_0\partial_0 + 2), \quad H = 2y\partial_y + x_0\partial_0 + x_1\partial_1 + 2 \\
E_{p^0} &= -3\sqrt{3}y\partial_0, \quad F_{p^0} = -\frac{2}{3\sqrt{3}}\left(x_0\partial_y - \frac{ix_1^3}{y^2}\right) \\
E_{q^0} &= -\frac{i}{3\sqrt{3}}x_0, \quad F_{q^0} = -\frac{2}{3\sqrt{3}}[-y\partial_1^3 + 27i(2 + x_0\partial_0 + x_1\partial_1 + y\partial_y)\partial_0] \\
E_{p^1} &= y\partial_1, \quad F_{p^1} = 2x_1\partial_y + \frac{4x_1^2\partial_1 + 4x_1}{3y} + \frac{2i}{9}x_0\partial_1^2 \\
E_{q^1} &= ix_1, \quad F_{q^1} = -\frac{6x_1^2}{y}\partial_0 + 2i\left(\frac{4}{3} + \frac{1}{3}x_1\partial_1 + x_0\partial_0 + y\partial_y\right)\partial_1 \\
Y_+ &= \frac{1}{\sqrt{2}}(iy\partial_1^2 + 9x_1\partial_0), \quad Y_- = -\frac{1}{\sqrt{2}}\left(-\frac{1}{3}x_0\partial_1 - i\frac{x_1^2}{y}\right) \\
F &= -\frac{2}{27}x_0\partial_1^3 - \frac{2}{y^2}x_1^3\partial_0 - \frac{2i}{9y}(3x_1^2\partial_1^2 + 6x_1\partial_1 + 2) - 2i(2 + y\partial_y + x_0\partial_0 + x_1\partial_1)\partial_y
\end{aligned} \tag{3.73}$$

It is easy to check that the principal symbols associated to these generators agree with the holomorphic moment maps (3.50), upon identifying

$$v^\flat = -2y, \quad v^0 = 2x_0, \quad v^1 = 2x_1, \quad w^\flat = -\frac{i}{2}p_y, \quad w_0 = \frac{i}{2}p_0, \quad w_1 = \frac{i}{2}p_1 \tag{3.74}$$

The minimal representation in the real polarization is unitary under the inner product

$$\langle f_1 | f_2 \rangle = \int dy dx_0 dx_1 f_1^*(y, x_0, x_1) f_2(y, x_0, x_1) \tag{3.75}$$

Alternatively, one may redefine $q_0 = -x_0/\sqrt{27y}$, $q_1 = x_1/\sqrt{y}$, $x = \sqrt{y}$ [58], leading to the polarization used by [29, 32],

$$\begin{aligned}
E &= -\frac{i}{2}x^2, \quad F = -\frac{i}{2}\partial_x^2 - \frac{3i}{2x}\partial_x + \frac{1}{18x^2}I_4 \\
E_{p^0} &= x\partial_0, \quad F_{p^0} = \frac{1}{9x}\left(2i\sqrt{3}q_1^3 - 9q_0(q_0\partial_0 + q_1\partial_1)\right) + q_0\partial_x \\
E_{p^1} &= x\partial_1, \quad F_{p^1} = \frac{1}{3x}\left(4q_1 + q_1^2\partial_1 + 2i\sqrt{3}q_0\partial_1^2 - 3q_0q_1\partial_0\right) + q_1\partial_x \\
E_{q^0} &= ixq_0, \quad F_{q^0} = \frac{1}{9x}\left(2\sqrt{3}\partial_1^3 + 9i(q_0\partial_0^2 + q_1\partial_0\partial_1 + 3\partial_0)\right) + i\partial_0\partial_x \\
E_{q^1} &= ixq_1, \quad F_{q^1} = \frac{1}{3x}\left(5i\partial_1 - iq_1\partial_1^2 + 2\sqrt{3}q_1^2\partial_0 + 3iq_0\partial_0\partial_1\right) + i\partial_1\partial_x \\
Y_+ &= \frac{i}{\sqrt{2}}\left(\partial_1^2 + i\sqrt{3}q_1\partial_0\right), \quad Y_- = \frac{1}{\sqrt{2}}\left(-iq_1^2 + \sqrt{3}q_0\partial_1\right) \\
Y_0 &= -\frac{1}{2}(3q_0\partial_0 + q_1\partial_1 + 2), \quad H = x\partial_x + 2
\end{aligned} \tag{3.76}$$

where

$$I_4(q_\Lambda, \partial_\Lambda) = 4\sqrt{3}q_0\partial_1^3 + 4\sqrt{3}q_1^3\partial_0 - 3iq_1^2\partial_1^2 + 18iq_0q_1\partial_0\partial_1 + 9iq_0^2\partial_0^2 + 3iq_1\partial_1 + 27iq_0\partial_0 - 8i \tag{3.77}$$

In accord with irreducibility, the quadratic Casimir evaluates to a constant

$$C_2 = -\frac{14}{9} \tag{3.78}$$

which coincides with that of the quaternionic discrete representation for $k = 4/3$ or $k = -14/3$. In addition, the minimal representation is annihilated by the Joseph ideal, e.g.

$$4Y_+^2 + 3\sqrt{3}(E_{p^1}F_{q^0} - E_{p^0}F_{q^1}) = 0 \tag{3.79a}$$

$$E_{p^1}^2 + \sqrt{3}E_{p^0}E_{q^1} - 2\sqrt{2}EY_+ = 0 \tag{3.79b}$$

$$E_{q^1}^2 - \sqrt{3}E_{q^0}E_{p^1} - 2\sqrt{2}EY_- = 0 \tag{3.79c}$$

$$3E_{p^0}E_{q^0} + E_{p^1}E_{q^1} - 4EY_0 = 0 \tag{3.79d}$$

These last three identities can be shown to imply the holomorphic anomaly equations satisfied by the topological amplitude in the one-modulus model with prepotential $F = -(X^1)^3/X^0$ [15]. Other identities will be discussed in Section 3.5.3 below.

For later reference, we note that the vector

$$f_P(y, x^0, x^1) = (x^0)^{-2/3} \exp\left[-i\frac{(x^1)^3}{yx^0}\right] \tag{3.80}$$

transforms as a one-dimensional representation of the Heisenberg parabolic P (more specifically, it is annihilated by $Y_+, Y_0, Y_-, F_{p^I}, F_{q_I}, F$ and carries charge $4/3$ under H). In particular, it is invariant under the Weyl reflection S with respect to the root E and therefore under Fourier transform over x^0, x^1 . The power $(P^0)^{-2/3}$ is consistent with the semi-classical analysis in [59].

3.5.2 Complex polarization

It is also useful to consider a different realization of the minimal representation [30], on functions $f(x, a^\dagger, b^\dagger)$ such that

$$R_0 = i \left(\frac{1}{2} a^\dagger a - \frac{3}{2} b^\dagger b - \frac{1}{2} \right), \quad R_+ = \frac{i}{\sqrt{2}} \left(a^\dagger a^\dagger + \sqrt{3} ab \right), \quad R_- = \frac{i}{\sqrt{2}} \left(a^2 + \sqrt{3} a^\dagger b^\dagger \right)$$

$$L_0 = \frac{i}{4} (x^2 - \partial_x^2) + \frac{i}{9x^2} I_4(a, b), \quad L_\pm = -\frac{i}{4\sqrt{2}} \left[(\partial_x \mp x)^2 + \frac{4}{9x^2} I_4 \right] \quad (3.81)$$

Here

$$I_4(a, b) = -\sqrt{3} (a^3 b + a^{3\dagger} b^\dagger) - \frac{9}{2} N_a N_b - \frac{3}{4} N_a^2 + \frac{9}{4} N_b^2 - 3N_a - \frac{41}{16} \quad (3.82)$$

where $a, b, a^\dagger, b^\dagger$ are bosonic oscillators with

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \quad N_a = a^\dagger a, \quad N_b = b^\dagger b \quad (3.83)$$

so that $a \equiv \partial_{a^\dagger}, b \equiv \partial_{b^\dagger}$; we shall refer to this presentation as the “upside-up” complex polarization. Alternatively, we could consider functions $f(x, a^\dagger, b)$ and represent $b^\dagger = -\partial_b$: this will be termed as the “upside-down” complex polarization, and will turn out to be the most convenient one to compute the lowest K -type. Irrespective of the choice of polarization for the oscillator algebra, the generators in the non-compact basis read

$$E = -\frac{i}{2} x^2, \quad F = -\frac{i}{2} \left(\partial_x^2 - \frac{4}{9x^2} I_4 \right) \quad (3.84)$$

$$E_{p^0} = \frac{ix}{2\sqrt{2}} \left[\sqrt{3}(a^\dagger + a) - (b^\dagger + b) \right], \quad E_{p^1} = -\frac{x}{2\sqrt{2}} \left[(a - a^\dagger) + \sqrt{3}(b - b^\dagger) \right]$$

$$E_{q_1} = -\frac{ix}{2\sqrt{2}} \left[(a^\dagger + a) + \sqrt{3}(b^\dagger + b) \right], \quad E_{q_0} = \frac{x}{2\sqrt{2}} \left[\sqrt{3}(a^\dagger - a) - (b^\dagger - b) \right]$$

while the compact generators are given by

$$J_+ = -\frac{i}{2} \left[\left(\partial_x - x - \frac{N_a - N_b + \frac{1}{2}}{x} \right) b - \frac{2}{3\sqrt{3}} \frac{a^{\dagger 3}}{x} \right] \quad (3.85a)$$

$$J_- = \frac{i}{2} \left[\left(\partial_x + x + \frac{N_a - N_b - \frac{1}{2}}{x} \right) b^\dagger + \frac{2}{3\sqrt{3}} \frac{a^3}{x} \right] \quad (3.85b)$$

$$S_+ = \frac{i\sqrt{3}}{2} \left[\left(\partial_x - x - \frac{\frac{1}{3}N_a + N_b + \frac{5}{6}}{x} \right) a - \frac{2}{\sqrt{3}} \frac{a^{\dagger 2} b^\dagger}{x} \right] \quad (3.85c)$$

$$S_- = -\frac{i\sqrt{3}}{2} \left[\left(\partial_x + x + \frac{\frac{1}{3}N_a + N_b + \frac{1}{2}}{x} \right) a^\dagger + \frac{2}{\sqrt{3}} \frac{a^2 b}{x} \right] \quad (3.85d)$$

This presentation of the minimal representation is related to (3.73) by a Bogoliubov transformation which is the quantum version of the canonical transformation (3.51). It can be decomposed into (i) a Fourier transform with respect to x_0

$$f(y, x_0, x_1) = \int dp_0 e^{-ip_0 x_0/y} f(y, p_0, x_1) \quad (3.86)$$

(ii) a change of variables

$$p_0 = -\frac{x\sqrt{3}}{18}(w - \sqrt{3}v), \quad x_1 = -\frac{x}{2}(\sqrt{3}w + v), \quad y = x^2, \quad (3.87)$$

and finally (iii) a standard Bogoliubov transform

$$f(x, v, w) = \int \exp \left[\frac{1}{2}[w^2 + (b^\dagger)^2 - v^2 - a^2] + \sqrt{2}(av - b^\dagger w) \right] f(x, a, b^\dagger) \quad (3.88)$$

implementing the change from real to oscillator polarization $a + a^\dagger = v\sqrt{2}$, $b + b^\dagger = w\sqrt{2}$. One may check that this sequence of unitary transformations in fact implements the Cayley transform (3.23).

3.5.3 K -type decomposition

Let us now discuss the K -type decomposition (3.72) of the minimal representation in more detail. We note that the correlation between the two spins is a straightforward consequence of the identity in the Joseph ideal²³

$$9C_2(J) - C_2(S) + 2 = 0. \quad (3.89)$$

The fact that the lowest K -type is a $SU(2)_J$ singlet and $SU(2)_S$ triplet²⁴ is less obvious, and will be further discussed below.

²³A similar identity holds in the non-compact basis, $9C_2(L) - C_2(R) + 2 = 0$.

²⁴This is an exception among quaternionic Lie groups; the lowest K -type of the minimal representation is usually a singlet of the Levi factor of P , while carrying a non-zero $SU(2)_J$ spin [26].

We also note that the K -type decomposition (3.72) is consistent with the known decomposition of the minimal representation under $SL(3, \mathbb{R})$ and $SU(2, 1)$ [8]: indeed, the minimal representation is an irreducible representation in the non-spherical principal series of $SL(3, \mathbb{R})$. The K -type decomposition of this representation, worked out in [5] (Eq. 7.8),

$$[1] + [2] + [3]^2 + [4]^2 + [5]^3 + [6]^3 + [7]^4 + [8]^4 + \dots \quad (3.90)$$

is consistent with the diagonal embedding of the maximal compact of $SL(3, \mathbb{R})$ inside $SU(2)_J \times SU(2)_S$. Under $SU(2, 1)$, the minimal representation decomposes as a sum of three irreducible principal series representations of $SU(2, 1)$ with infinitesimal characters $(0, \frac{1}{3}, -\frac{1}{3}), (\frac{1}{3}, 0, -\frac{1}{3}), (\frac{1}{3}, -\frac{1}{3}, 0)$. These three representations correspond to the three supplementary series at $p = q = -2/3$ in the terminology of [6], and transform with different characters of the center \mathbb{Z}_3 of $SU(2, 1)$. The K -type decomposition of these representations is given on Figure 6, and is also in agreement with (3.72). We also compute the Casimirs,

$$C_2[SL(3, \mathbb{R})] = -\frac{8}{9}, \quad C_3[SL(3, \mathbb{R})] = 0 \quad (3.91a)$$

$$C_2[SU(2, 1)] = -\frac{8}{9}, \quad C_3[SU(2, 1)] = 0, \quad p = q = -2/3 \quad (3.91b)$$

Let us now further analyze (3.89), by rewriting them in terms of the generators of G acting in the minimal representation. Using

$$C_2(J) = -J_3^2 - \frac{1}{2}(J_+J_- + J_-J_+) = -J_3(J_3 \pm i) - J_{\mp}J_{\pm} \quad (3.92a)$$

$$C_2(S) = -S_3^2 - \frac{1}{2}(S_+S_- + S_-S_+) = -S_3(S_3 \pm i) - S_{\mp}S_{\pm} \quad (3.92b)$$

we see that a normalizable eigenmode f of J_3, S_3 satisfying the highest weight condition $J_+ f = S_+ f = 0$ necessarily has $(J_3, S_3) \in \frac{i}{2}\mathbb{N}$. From (3.89), we have

$$9J_3(J_3 + i) - S_3(S_3 + i) + 2 = 4(2R_0 + i)(\tilde{R}_0 + i) = 0 \quad (3.93)$$

where

$$R_0 = \frac{1}{2}(3J_3 - S_3), \quad \tilde{R}_0 = \frac{1}{2}(3J_3 + S_3) \quad (3.94)$$

Thus it is either an eigenmode of $R_0 = -i/2$, or of $\tilde{R}_0 = -i$. The second option is inconsistent with the highest weight condition, so $R_0 = -i/2$. With similar arguments

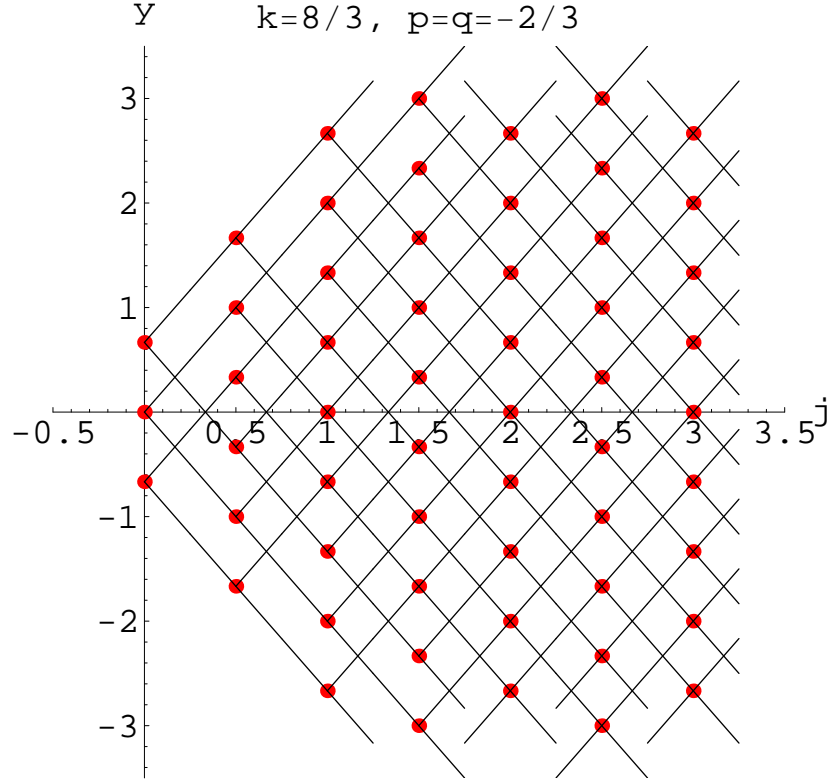


Figure 6: K -type decomposition of the three supplementary series representations of $SU(2, 1)$ at $p = q = -2/3$, in the $(j, y = 2/3s)$ plane.

we conclude that

$$J_+ f = S_+ f = 0 \quad \Rightarrow \quad J_3 = \frac{i}{2}m, \quad S_3 = \frac{i}{2}(3m + 2), \quad R_0 = -\frac{i}{2} \quad (3.95a)$$

$$J_+ f = S_- f = 0 \quad \Rightarrow \quad J_3 = \frac{i}{2}m, \quad S_3 = -\frac{i}{2}(3m + 2), \quad \tilde{R}_0 = -\frac{i}{2} \quad (3.95b)$$

$$J_- f = S_+ f = 0 \quad \Rightarrow \quad J_3 = -\frac{i}{2}m, \quad S_3 = \frac{i}{2}(3m + 2), \quad \tilde{R}_0 = \frac{i}{2} \quad (3.95c)$$

$$J_- f = S_- f = 0 \quad \Rightarrow \quad J_3 = -\frac{i}{2}m, \quad S_3 = -\frac{i}{2}(3m + 2), \quad R_0 = \frac{i}{2} \quad (3.95d)$$

In the following, we analyze the consequences of these equations in various polarizations.

3.5.4 Lowest K -type in the complex polarization

It turns out that the form of the lowest K -type is simplest in the complex polarization,

where the generator R_0 takes a simple form

$$R_0 = i \left(\frac{1}{2} a^\dagger a - \frac{3}{2} b^\dagger b - \frac{1}{2} \right) \quad (3.96)$$

The following linear combinations of (3.85) lead to x -independent equations:

$$T_{++} = a\sqrt{3}J_+ + bS_+ = -\frac{i}{x}(a^{\dagger 2} + \sqrt{3}ab)(R_0 + \frac{i}{2}) \quad (3.97a)$$

$$T_{--} = \sqrt{3}a^\dagger J_- + b^\dagger S_- = -\frac{2}{3}x(a^2 + \sqrt{3}a^\dagger b^\dagger)(R_0 - \frac{i}{2}) \quad (3.97b)$$

$$T_{+-} = a^\dagger \sqrt{3}J_+ - bS_- , \quad T_{-+} = a \sqrt{3}J_- - b^\dagger S_+ \quad (3.97c)$$

where the first two lines are consistent with (3.95a).

Now we restrict to a highest weight $S_+ f = 0$, which is a singlet of $SU(2)_J$, i.e. $J_+ f = J_- f$. It turns out that it is convenient to work in the “downside-up polarization” where $b = \partial_{b^\dagger}$, $a^\dagger = -\partial_a$. The constraint $R_0 = -i/2$ requires

$$f(x, a, b^\dagger) = (b^\dagger)^{-1/3} f_1(x, z) , \quad z = \frac{1}{3\sqrt{3}} \frac{a^3}{b^\dagger} \quad (3.98)$$

The constraint T_{++} is automatically obeyed, however T_{-+} leads to a second order ordinary differential equation in z only,

$$\left[z\partial_z^2 - 2(z - \frac{1}{3})\partial_z + (z + x^2 - \frac{2}{3}) \right] f_1(x, z) = 0 \quad (3.99)$$

The solution is

$$f_1(x, z) = x^{1/3} z^{1/6} e^z \left[K_{1/3}(2ix\sqrt{z}) f_2(x) + I_{1/3}(2ix\sqrt{z}) \tilde{f}_2(x) \right] \quad (3.100)$$

In the following, we assume that normalizability forces $\tilde{f}_2(x) = 0$. It is one of the drawbacks of the complex polarization that normalizability is difficult to check – at any rate, it is straightforward to generalize the computation below to include both solutions. Requiring the action of S_+ on (3.98) to vanish, we find

$$\left[6x\partial_x - 12z\partial_z^2 + 2(6z - 4)\partial_z + (1 - 6x^2) \right] f_1(x, z) = 0 \quad (3.101)$$

Combining this with (3.99), we can produce a first order partial differential equation

$$\left[x\partial_x - 2z\partial_z + \left(2z + y^2 - \frac{7}{6} \right) \right] f_1(x, z) = 0 \quad (3.102)$$

whose solution is

$$f_1(x, z) = x^{7/6} e^{-x^2/2+z} f_3(x^2 z) \quad (3.103)$$

This uniquely determines

$$f_2(x) = x^{-7/6} e^{-x^2/2}, \quad f_3(u) = K_{1/3}(2i\sqrt{u}) \quad (3.104)$$

It is now easy to check that the resulting function is annihilated by S_+ , J_+ , J_- , J_3 , and is an eigenmode of $S_3 = i$. The remaining two states in the triplet may be obtained by acting with S_- . Altogether, we find that the lowest K -type is

$$f_{0,1} = (ax^3/b^\dagger)^{1/2} e^{z-x^2/2} K_{1/3}(2ix\sqrt{z}) \quad (3.105a)$$

$$f_{0,0} = 2 \cdot 3^{-1/4} x^{3/2} a (b^\dagger)^{-1} e^{z-x^2/2} K_{2/3}(2ix\sqrt{z}) \quad (3.105b)$$

$$f_{0,-1} = -\frac{2}{\sqrt{3}} (ax/b^\dagger)^{3/2} e^{z-x^2/2} K_{1/3}(-2ix\sqrt{z}) \quad (3.105c)$$

The highest weights in the higher K -types can be obtained by acting with the raising operators $J_{1/2,3/2}$. For example,

$$f_{\frac{1}{2},\frac{5}{2}} = \frac{1}{3} x^{3/2} a^{1/2} (b^\dagger)^{-1} e^{z-x^2/2} \left[K_{\frac{4}{3}} + i \cdot 3^{\frac{1}{4}} a^{\frac{3}{2}} x K_{\frac{2}{3}} + 3\sqrt{b^\dagger} (2x^2 - 3) K_{\frac{1}{3}} \right] \quad (3.106)$$

where the argument of the Bessel function is as in $f_{0,1}$. We note that semi-classically, all K -types behave as

$$\exp \left[-\frac{x^2}{2} + \frac{1}{3\sqrt{3}} \frac{a^3}{b^\dagger} + \frac{2}{3^{3/4}} ix \sqrt{\frac{a^3}{b^\dagger}} \right] \quad (3.107)$$

As explained in (2.6.2), and further at the end of the next subsection, the argument of the exponential (or “classical action”) provides the generating function for a complex Lagrangian cone inside the hyperkähler cone \mathcal{S} .

3.5.5 Lowest K -type in the real polarization

According to the above, the lowest K -type should be a triplet of $SU(2)_S$, singlet under $SU(2)_J$. Thus we impose the conditions

$$J_+ = J_3 = J_- = S_+ = 0 \quad (3.108)$$

Note in particular that the generator $F_{p^0} - E_{p^0}$ of rotations in the (y, x_0) plane, which was as the start of the KPW solution for the spherical vector in the split case, no longer annihilates the state, so we need a different strategy. Our approach is to find a linear combination of the operators J_+ , J_3 , J_- , S_+ which involves first order differential operators only. The one of interest is

$$S_3 - 3J_3 + 3i\sqrt{\frac{3}{2}} \frac{y}{x_0} (J_+ - 2J_- - S_+) \quad (3.109)$$

which allows to rewrite S_3 as a first order differential operator in three variables,

$$S_3 = \alpha_y \partial_y + \alpha_0 \partial_0 + \alpha_1 \partial_1 + \beta \quad (3.110)$$

where

$$\alpha_y = y \left(i - 9 \frac{x_1}{x_0} \right), \quad \alpha_0 = -9x_1 - \frac{27i}{2x_0} y^2, \quad \alpha_1 = \frac{x_0}{3} + \frac{-12x_1^2 + 9y^2}{2x_0}, \quad (3.111)$$

$$\beta = \frac{x_1(-6y + x_1^2 + ix_0x_1)}{x_0y} \quad (3.112)$$

The standard way to solve this equation is to integrate the flow

$$\frac{dy}{ds} = \alpha_y, \quad \frac{dx_0}{ds} = \alpha_0, \quad \frac{dx_1}{ds} = \alpha_1, \quad \frac{df}{ds} = (\beta - s_3)f \quad (3.113)$$

Using inspiration from the split case [39], it is easy find one constant of motion along the flow,

$$z = \frac{(y^2 + \frac{2}{27}x_0^2 + \frac{2}{3}x_1^2)^{3/2}}{y^2 + \frac{2}{27}x_0^2} \quad (3.114)$$

To find the second constant of motion, and integrate the flow completely, we go to polar coordinates in the (y, x_0) plane,

$$y = \sqrt{2} r \cos \theta, \quad x_0 = 3\sqrt{3} r \sin \theta \quad (3.115)$$

We now change variables to

$$f(y, x_0, x_1) = r^{-2/3} h(z, r, \theta) \exp \left(-2i \frac{x_0 x_1^3}{y(2x_0^2 + 27y^2)} \right) \quad (3.116)$$

The action of S_3 on $h(z, r, \theta)$ is still a first order differential operator, but now involving only two variables r, θ :

$$S_3 = -i \cot \theta \partial_\theta - \frac{3r}{\sin \theta} \left[\left(\frac{z}{r\sqrt{2}} \right)^{2/3} - 1 \right]^{1/2} \partial_r \quad (3.117)$$

Again, the way to solve this equation is to integrate the flow

$$\frac{d\theta}{ds} = -\frac{i}{\tan \theta}, \quad \frac{dr}{ds} = -\frac{3r}{\sin \theta} \left[\left(\frac{z}{r\sqrt{2}} \right)^{2/3} - 1 \right]^{1/2}, \quad \frac{dh}{hds} = s_3 \quad (3.118)$$

The ratio of the first two equations gives $d\theta/dr$, and produces a second constant of motion (in addition to z),

$$t = \frac{(1 - i e^{i\theta})^2 \left(1 + i \sqrt{\left(\frac{z}{r\sqrt{2}} \right)^{2/3} - 1} \right)}{(1 + i e^{i\theta})^2 \left(1 - i \sqrt{\left(\frac{z}{r\sqrt{2}} \right)^{2/3} - 1} \right)} \quad (3.119)$$

The third equation then constrain the θ dependence to

$$h(z, r, \theta) = [\cos \theta]^{-is_3} h_1(z, t) \quad (3.120)$$

where s_3 is the eigenvalue under S_3 , which we independently know to be $s_3 = i$ for the lowest K -type.

We now express the action of the other generators J_{\pm}, S_{+}, R_0 on our ansatz

$$f(y, x_0, x_1) = \cos \theta r^{-2/3} h_1(z, r) \exp \left(-2i \frac{x_0 x_1^3}{y(2x_0^2 + 27y^2)} \right) \quad (3.121)$$

For this purpose, we express r, θ, z, t in terms of y, x_0, x_1 using (3.114), (3.115) and

$$t = \left[1 + \frac{4}{27} \frac{x_0^2}{y^2} + \frac{2\sqrt{2}}{27} \frac{x_0}{y^2} \sqrt{2x_0^2 + 27y^2} \right] \frac{1 + \frac{3i\sqrt{2}x_1}{\sqrt{2x_0^2 + 27y^2}}}{1 - \frac{3i\sqrt{2}x_1}{\sqrt{2x_0^2 + 27y^2}}}, \quad (3.122)$$

and act with the original differential operators. We then set $x_1 = 0$, and revert to z, t variables using

$$x_0 = \sqrt{27(r^2 - \frac{1}{2}y^2)}, \quad r = \frac{z}{\sqrt{2}}, \quad y = \frac{2\sqrt{t}}{1+t} z \quad (3.123)$$

The last two identities are only valid at $x_1 = 0$, but that is sufficient to get the full action on $h_1(z, r)$. We find, in particular,

$$\begin{aligned} R_0 + \frac{i}{2} &= -\frac{i\sqrt{t}}{9(t+1)^2 z^{5/3}} \left(2t \left(3tz - 3z + 8\sqrt{t} \right) \partial_t + 4\sqrt{t} (4t^2 \partial_t^2 - 3z \partial_z) - 3(t+1)z \right) \\ J_+ + J_- &= \frac{t}{18(t+1)z^{5/3}} \left(2 \left(3tz + 3z + 4\sqrt{t} \right) \partial_t - 24\sqrt{t} z \partial_z \partial_t - 3(t-1)z \right) \end{aligned} \quad (3.124)$$

A lengthy but straightforward analysis allows to determine the expansion of the solution at $z \rightarrow \infty$,

$$\begin{aligned} h_1(z, t) &\sim \frac{(1 + \sqrt{t})^2}{\sqrt{t}} z^{1/6} e^{-z/2} \left[1 + \left(\frac{2\sqrt{t}}{3(1 + \sqrt{t})^2} - \frac{5}{36} \right) z^{-1} \right. \\ &\quad \left. + \left(-\frac{35\sqrt{t}}{54(1 + \sqrt{t})^2} + \frac{385}{2592} \right) z^{-2} + \left(\frac{5005\sqrt{t}}{3878(1 + \sqrt{t})^2} + \frac{-85085}{279936} \right) z^{-3} + \dots \right] \end{aligned} \quad (3.125)$$

This suggests the Ansatz

$$h_1(z, t) = z^{1/6} \left[h_2(z) + \frac{(1 + \sqrt{t})^2}{\sqrt{t}} h_3(z) \right] \quad (3.126)$$

Indeed, we find that all constraints reduce to two ordinary differential equations for h_2 and h_3 ,

$$(1 + 3z)h_2 + 6z(h'_2 + 4h'_3 + 2h_3) = 0 \quad (3.127a)$$

$$(4 + 15z)h_2 + (54z - 2)h_3 + 12z(2h'_2 + 9h'_3) = 0 \quad (3.127b)$$

which decouple into

$$(1 + 18z + 9z^2)h_2 - 36z(h'_2 + zh''_2) = 0 \quad (3.128a)$$

$$(9z^2 - 5)h_3 - 36z^2h''_3 = 0 \quad (3.128b)$$

Each of them has two independent solutions, but we keep only the one decaying at $z \rightarrow \infty$ in order to ensure normalizability, leading to

$$h_1(z, t) = \frac{(1 + \sqrt{t})^2}{\sqrt{t}} z^{2/3} K_{1/3}(z/2) + \frac{2\sqrt{\pi}}{3} z^{1/3} U_{\frac{7}{6}, \frac{4}{3}, 7}(z) \quad (3.129)$$

which correctly reproduces the subleading terms in (3.125). In total, the final answer for the lowest K -type in the real polarization is

$$\begin{aligned} f_{0,1}(y, x_0, x_1) = & (\cos \theta) r^{-2/3} \left[\frac{(1 + \sqrt{t})^2}{\sqrt{t}} z^{2/3} K_{1/3}(z/2) + \frac{2\sqrt{\pi}}{3} z^{1/3} U_{\frac{7}{6}, \frac{4}{3}, 7}(z) \right] \\ & \times \exp \left(-2i \frac{x_0 x_1^3}{y(2x_0^2 + 27y^2)} \right) \end{aligned} \quad (3.130)$$

In the semi-classical limit, where y, x_0, x_1 are scaled uniformly to ∞ , it reduces to

$$f_{0,1}(y, x_0, x_1) \sim r^{-2/3} \frac{(1 + \sqrt{t})^2}{\sqrt{t}} z^{1/6} \cos \theta e^{-S} \quad (3.131)$$

where S is the “classical action”

$$S = \frac{\left(\frac{2x_0^2}{27} + \frac{2x_1^2}{3} + y^2 \right)^{3/2}}{2 \left(\frac{2x_0^2}{27} + y^2 \right)} - \frac{2ix_0 x_1^3}{y(2x_0^2 + 27y^2)} \quad (3.132)$$

The same reasoning as in Section 2.6.2 implies that S is the generating function of a holomorphic Lagrangian cone inside the hyperkähler cone \mathcal{S} , invariant under the

maximal compact K . The precise identification of the variables y, x_0, x_1 in the minimal representation and the complex coordinates on \mathcal{S} was given in (3.74). One may indeed check that the holomorphic moment maps of the compact generators vanish identically on the locus $w_I = \partial_{v_I} S$.

After rescaling $x_0 \rightarrow x_0 \sqrt{27/2}, x_1 \rightarrow x_1/\sqrt{2}$, we find that S can be cast in the same form as in Eq. 4.72 of [39],

$$S = \frac{1}{2} \|X\|^2 - i \frac{x_0}{y \sqrt{x_0^2 + y^2}} \mathcal{F} \quad (3.133)$$

where X is an $Sp(2n+4)$ extension of the usual $Sp(2n+2)$ symplectic section appearing in the special geometry description of the c -map,

$$X = (y, x_0, x_1; \tilde{y}, \tilde{x}^0, \tilde{x}^1) \quad , \quad \|X\|^2 = y^2 + x_0^2 + x_1^2 + \tilde{y}^2 + (\tilde{x}^0)^2 + (\tilde{x}^1)^2 \quad (3.134)$$

$$\tilde{y} = \partial_y \mathcal{F} \quad , \quad \tilde{x}^i = \partial_{x^i} \mathcal{F} \quad , \quad \mathcal{F} = \frac{x_1^3}{3\sqrt{3}\sqrt{y^2 + x_0^2}} \quad (3.135)$$

It would be interesting to see whether such a $Sp(2n+4)$ invariant description also exists for non-symmetric c -map spaces, or even for general hyperkähler manifolds, and to investigate whether the lowest K -type of the minimal polarization bears any relation to the topological string amplitude of the corresponding magical supergravity theory as discussed in [15].

3.6 Small submodules of the principal series

In this section, we study the construction of “small” submodules of the principal series representation of $G_{2(2)}$ by imposing constraints directly, similarly to the discussion of the minimal representation for $SU(2, 1)$ in Section 2.5.3.

3.6.1 The minimal submodule

We start by considering the submodule of the principal series annihilated by the “holomorphic anomaly” relations

$$C_+ \equiv E_{p^1}^2 + \sqrt{3} E_{p^0} E_{q_1} - 2\sqrt{2} E Y_+ = 0 \quad (3.136a)$$

$$C_- \equiv E_{q_1}^2 - \sqrt{3} E_{q_0} E_{p^1} - 2\sqrt{2} E Y_- = 0 \quad (3.136b)$$

in the Joseph ideal (see (3.79b),(3.79c)). In terms of the differential operator realization (3.43), these conditions reduce to

$$-3(P^1)^2 + P^0 Q_1 = 0 \quad (3.137a)$$

$$(Q_1)^2 + 9Q_0 P^1 = 0 \quad (3.137b)$$

where

$$P^I = \partial_{\tilde{\zeta}_I} - \zeta^I \partial_\sigma, \quad Q_I = \partial_{\zeta^I} + \tilde{\zeta}_I \partial_\sigma, \quad (3.138)$$

are covariant derivatives commuting with E_{p^I}, E_{q_I} , analog of $\nabla, \bar{\nabla}$ in (2.128). For $k = 4/3$, this subspace is invariant under the action of G ; indeed, the commutators of the constraints with the lowest negative root F can be rewritten as

$$[F, C_+] = -2(\sigma + \frac{2}{3}D_+)C_0 + \frac{2}{3}D_0C_+ \quad (3.139a)$$

$$[F, C_-] = -2(\sigma - \frac{2}{9}D_-)C_0 - \frac{2}{3}D_0C_- \quad (3.139b)$$

where D_0, D_\pm is the $SL(2)$ triplet

$$D_+ = 3(\zeta^1)^2 + \zeta^0 \tilde{\zeta}_1, \quad D_0 = 3\zeta^0 \tilde{\zeta}_0 + \zeta^1 \tilde{\zeta}_1, \quad D_- = (\tilde{\zeta}_1)^2 - 9\zeta^1 \tilde{\zeta}_0 \quad (3.140)$$

At the level of differential symbols, the constraints (3.137) are solved by

$$Q_0 = -(P^1)^3/(P^0)^2, \quad Q_1 = 3(P^1)^2/(P^0) \quad (3.141)$$

It is therefore natural to go to a polarization where P^I and E act diagonally,

$$f(\zeta, \tilde{\zeta}, \sigma) = \int dp dK \exp\left(-iK\sigma - ip^I \tilde{\zeta}_I\right) g(\zeta^I, p^I, K) \quad (3.142)$$

In this polarization, the generators become

$$\begin{aligned} E &= iK, \quad Y_0 = \frac{1}{2}(3p^0 \partial_{p^0} + 3\zeta^0 \partial_{\zeta^0} + p^1 \partial_{p^1} + \zeta^1 \partial_{\zeta^1} + 4) \\ E_{p^0} &= -i(p^0 + K\zeta^0), \quad E_{q_0} = -\partial_{\zeta^0} - K\partial_{p^0} \\ E_{p^1} &= -i\sqrt{3}(p^1 + K\zeta^1), \quad E_{q_1} = -(\partial_{\zeta^1} + K\partial_{p^1})/\sqrt{3} \\ Y_+ &= \frac{1}{\sqrt{2}}(6ip^1 \zeta^1 + p^0 \partial_{p^1} + \zeta^0 \partial_{\zeta^1}), \quad Y_- = -\frac{1}{3\sqrt{2}}(9p^1 \partial_{p^0} + 2i\partial_{p^1} \partial_{\zeta^1} + 9\zeta^1 \partial_{\zeta^0}), \dots \end{aligned} \quad (3.143)$$

while the constraints are

$$C_+ = i(p^0 - K\zeta^0)(\partial_{\zeta^1} - K\partial_{p^1}) - 3(p^1 - K\zeta^1)^2 \quad (3.144a)$$

$$C_- = -3i(p^1 - K\zeta^1)(\partial_{\zeta^0} - K\partial_{p^0}) + \frac{1}{3}(\partial_{\zeta^1} - K\partial_{p^1})^2 \quad (3.144b)$$

An invariant set of solutions can be found by restricting to functions

$$g(\zeta^I, p^I, K) = (P^0 - 2K\zeta^0)^{-2/3} \exp\left[i \frac{(P^1 - 2K\zeta^1)^3}{2K(P^0 - 2K\zeta^0)}\right] h(P^I, K) \quad (3.145)$$

where $P^I = p^I + K\zeta^I$. The quasi-conformal action on $f(\zeta^I, \tilde{\zeta}_I, \sigma)$ at degree $k = 4/3$ restricts to an action on $h(P^I, K)$ given by

$$\begin{aligned} E &= -\frac{i}{2}y, \quad E_{p^0} = -\frac{i}{3\sqrt{3}}x^0, \quad E_{q_0} = 3\sqrt{3}y\partial_0, \quad E_{p^1} = ix^1, \quad E_{q_1} = -y\partial_1, \\ H &= 2y\partial_y + x^0\partial_0 + x^1\partial_1 + 2, \quad Y_0 = \frac{1}{2}(3x^0\partial_0 + x^1\partial_1 + 2) \\ Y_+ &= -\frac{1}{3\sqrt{2}}\left[x^0\partial_1 + 3i\frac{(x^1)^2}{y}\right], \quad Y_- = \frac{1}{\sqrt{2}}[iy\partial_1^2 + 9x^1\partial_0], \dots \end{aligned} \quad (3.146)$$

where we have redefined

$$x^0 = 3\sqrt{3}P^0, \quad x^1 = -\sqrt{3}P^1, \quad y = -2K \quad (3.147)$$

We recognize this as the minimal representation in the polarization (3.73), after applying a Weyl reflection S with respect to the highest root E , which has the effect of exchanging E_{p^I} with E_{q_I} and Y_+ with Y_- .

We conclude that the minimal representation can be embedded into the principal series at $k = 4/3$, via

$$\begin{aligned} f(\zeta^I, \tilde{\zeta}_I, \sigma) &= \int dP^I dK f_P^*(p^I - K\zeta^I, K) h(p^I + K\zeta^I, K) e^{-iK(\sigma + \zeta^I \tilde{\zeta}_I) - ip^I \tilde{\zeta}_I} \\ &= \langle f_P | e^{-\sigma E - \zeta^I E_{q_I} + \tilde{\zeta}_I E_{p^I}} | h \rangle \end{aligned} \quad (3.148)$$

where

$$f_P(K, P^0, P^1) = (P^0)^{-2/3} \exp\left[-i\frac{(P^1)^3}{2K(P^0)}\right] \quad (3.149)$$

is the P -covariant vector introduced in (3.80).

3.6.2 Intermediate representations

Above we described two representations π'_1 and $\pi'_{2/3}$ which we suggested (following the pattern of [26]) should be obtained as submodules of the principal series for $k = 1$ and $k = 2/3$, respectively. Now we verify that one can indeed find invariant subspaces by imposing the appropriate constraints at these values of k .

Keeping the same notation as in (3.138), we find that

$$\begin{aligned} I_4 &= \frac{4}{27}P^0(Q_1)^3 - 4Q_0(P^1)^3 + (Q_0)^2(P^0)^2 - \frac{1}{3}(Q_1)^2(P^1)^2 + 2Q_0P^0Q_1P^1 \\ &\quad - 6Q_0P^0 - \frac{2}{3}Q_1P^1 - \frac{5}{9}E^2 \end{aligned} \quad (3.150)$$

commutes with all generators for $k = 1$, in particular

$$[F, I_4] = -4E I_4 \quad (3.151)$$

Imposing the constraint $I_4 = 0$ on the representation space of a principal series representation we expect to find the representation π'_1 with Gelfand-Kirillov dimension 4.

Similarly, we find that the constraints²⁵

$$\partial^0 I_4 \equiv \partial I_4 / \partial Q_0 = -4(P^1)^3 + 2Q_0(P^0)^2 + 2P^0 Q_1 P^1 - \frac{32}{3} E P^0 \quad (3.152a)$$

$$\partial^1 I_4 \equiv \partial I_4 / \partial Q_1 = \frac{4}{9} P^0 Q_1^2 - \frac{2}{3} Q_1 (P^1)^2 + 2Q_0 P^0 P^1 - \frac{16}{3} E P^1 \quad (3.152b)$$

$$\partial_1 I_4 \equiv \partial I_4 / \partial P^1 = -12Q_0 (P^1)^2 - \frac{2}{3} Q_1^2 P^1 + 2Q_0 P^0 Q_1 + 4E Q_1 \quad (3.152c)$$

$$\partial_0 I_4 \equiv \partial I_4 / \partial P^0 = \frac{4}{27} Q_1^3 + 2Q_0^2 P^0 + 2Q_0 Q_1 P^1 - \frac{4}{3} E Q_0 \quad (3.152d)$$

commute with the generators for $k = 2/3$. In particular,

$$[F, \partial^0 I_4] = (D_0 - 3\alpha) \partial^0 I_4 - 2D_+ \partial^1 I_4 \quad (3.153a)$$

$$[F, \partial^1 I_4] = (\frac{1}{3} D_0 - 3\alpha) \partial^1 I_4 - \frac{4}{9} D_+ \partial_1 I_4 - \frac{2}{9} D_- \partial^0 I_4 \quad (3.153b)$$

3.7 3-step radical and 7-grading

All the constructions in this paper so far hinged on the existence of Heisenberg parabolic P with unipotent radical N of order 2, and the resulting 5-grading of \mathfrak{g} . In section 3.7.1, we construct a representation of G which relies on the 7-grading by the Cartan generator Y_0 (the vertical axis in Figure 4, left), corresponding to a parabolic subgroup P_3 with unipotent radical N_3 of degree 3. In Section 3.7.2, we give a new polarization of the minimal representation appropriate to this parabolic subgroup. Physically, we expect these representations to become relevant in describing black holes in five dimensions.

3.7.1 Induced representation from 3-step radical

It is possible to construct a different representation using the 7-grading by the non-compact generator Y_0 : we decompose

$$g = p_3 \cdot n_3 = p_3 \cdot e^{\sqrt{\frac{3}{2}}(mF_{q^1} + nE_{p^1})} \cdot e^{\sqrt{2}tY_+} \cdot e^{\frac{1}{\sqrt{2}}(kF_{q^0} + lE_{p^0})} \quad (3.154)$$

²⁵While $dI_4 = 0$ implies $I_4 = 0$ classically (at the level of the moment maps), this is no longer true quantum mechanically; in fact $Q_I \partial^I I_4 + P^I \partial_I I_4 = 4I_4 - \frac{28}{9}$.

where p is an element of the parabolic subgroup P_3 opposite to the 3-step nilpotent N_3 . Acting on functions $f(m, n, t, k, l)$ transforming with a character $\exp(jY_0)$ of P , we find

$$\begin{aligned}
F_{q_0} &= \sqrt{2}\partial_k, \quad E_{p^0} = \sqrt{2}\partial_l, \quad Y_+ = \frac{1}{\sqrt{2}}\partial_t, \quad E = k\partial_l + m\partial_n, \quad F = l\partial_k + n\partial_m \\
F_{q^1} &= \sqrt{\frac{2}{3}} [\partial_m - n\partial_t + (3t - mn)\partial_k - n^2\partial_l] \\
E_{p^1} &= \sqrt{\frac{2}{3}} [\partial_n + m\partial_t + (3t + mn)\partial_l + m^2\partial_k] \\
H &= m\partial_m - n\partial_n + k\partial_k - l\partial_l, \quad Y_0 = -\frac{1}{2}(m\partial_m + n\partial_n) - t\partial_t - \frac{3}{2}(k\partial_k + l\partial_l) + j \\
E_{q_1} &= -\sqrt{\frac{2}{3}} [m^2\partial_m + (2t + mn)\partial_n + (2mt - k)\partial_t + 2m^2t\partial_k + t(3t + 2mn)\partial_l - jm] \\
F_{p^1} &= \sqrt{\frac{2}{3}} [(2t - mn)\partial_m - n^2\partial_n + (l - 2nt)\partial_t + t(3t - 2mn)\partial_k - 2n^2t\partial_l + jn] \\
Y_- &= \frac{1}{\sqrt{2}} [(k + tm)\partial_m + (l + tn)\partial_n + (t^2 + lm - kn)\partial_t \\
&\quad + [m(lm - kn) + 3kt]\partial_k + [n(lm - kn) + 3lt]\partial_l - 2jt] \\
F_{p^0} &= -\sqrt{2} \{ [(k - tm)n + t^2]\partial_m + n(l - tn)\partial_n + t(l - tn)\partial_t \\
&\quad + [kl + t^2(t - mn)]\partial_k + (l^2 - n^2t^2)\partial_l - j(l - nt) \} \\
E_{q^1} &= -\sqrt{2} \{ m(k - tm)\partial_m + [(l - tn)m + t^2]\partial_n + t(k - tm)\partial_t \\
&\quad + (k^2 - m^2t^2)\partial_k + [kl - t^2(t + mn)]\partial_l + j(l - nt) \} \tag{3.155}
\end{aligned}$$

The quadratic Casimir is

$$C_2 = \frac{1}{3}j(j + 5) \tag{3.156}$$

The spherical vector is easy to determine: in the fundamental representation, the first row of g transforms in a 1-dimensional representation of the parabolic P . Hence,

$$f_K = [1 + m^2 + n^2 + 2t^2 + (k - mt)^2 + (l - nt)^2 + (t^2 + kn - lm)]^{j/2} \tag{3.157}$$

3.7.2 A 3-step polarization for the minimal representation

We now present a third polarization for the minimal representation, obtained by diagonalizing the generators Y_-, E_{q^0}, F_{p^0} , which correspond to the height 2 and 3 generators of the 3-step nilpotent. We shall denote by iQ, iL, iJ the corresponding eigenvalues. In the real polarization (3.76), the eigenmodes of these three commuting generators are given by

$$f_{Q,J,L}(q_0, q_1, x) = \exp \left[i \left(\frac{q_1^3}{3\sqrt{3}q_0} + \frac{\sqrt{6}}{3}Q\frac{q_1}{q_0} + J\frac{x}{2q_0} \right) \right] \delta(q_0x - L) \tag{3.158}$$

Thus, we set

$$f(q_0, q_1, x) = \int dQ dJ \exp \left[i \left(\frac{q_1^3}{3\sqrt{3}q_0} + \frac{\sqrt{6}}{3} Q \frac{q_1}{q_0} + J \frac{L}{2q_0^2} \right) \right] g(Q, J, L = q_0 x) \quad (3.159)$$

The 3-step polarization is therefore related to the polarization (3.73) by multiplying by $e^{F(q_i)}$, and Fourier transforming over $q_1/q_0 \rightarrow Q$ and $1/q_0^2 \rightarrow J/L$. The resulting generators are

$$\begin{aligned} E &= L\partial_J, \quad Y_- = iQ, \quad F_{p^0} = iJ, \quad E_{q_0} = iL \\ E_{p^0} &= 2i(3\partial_J + L\partial_J\partial_L + J\partial_J^2 + Q\partial_Q\partial_J) - \frac{L}{2\sqrt{2}}\partial_Q^3 \\ E_{p^1} &= -2\sqrt{\frac{2}{3}}Q\partial_J - i\frac{\sqrt{3}}{2}L\partial_Q^2, \quad E_{q_1} = -\sqrt{\frac{3}{2}}L\partial_Q \\ Y_+ &= -\frac{i}{2}(Q\partial_Q^2 + 3J\partial_Q\partial_J + L\partial_Q\partial_L + 7\partial_Q)\frac{2\sqrt{2}}{3L}Q^2\partial_J \\ Y_0 &= -\frac{1}{2}(2Q\partial_Q + 3J\partial_J + 3L\partial_L + 7), \quad H = L\partial_L - J\partial_J + 1 \\ F_{p_1} &= -\frac{\sqrt{6}}{2}J\partial_Q - \frac{4\sqrt{3}}{9L}Q^2, \quad F_{q_1} = i\frac{\sqrt{3}}{2}J\partial_Q^2 - \frac{4}{3}\sqrt{\frac{2}{3}}\frac{Q^2}{L}\partial_Q - 2\sqrt{\frac{2}{3}}Q\partial_L - \frac{10}{3}\sqrt{\frac{2}{3}}\frac{Q}{L} \\ F_{q_0} &= \frac{1}{2\sqrt{2}}\partial_Q^3 + \frac{8\sqrt{2}}{27L^2}Q^3\partial_J + \frac{2i}{3L}Q^2\partial_Q^2 + 2iQ\partial_Q\partial_L + \frac{10i}{3L}Q\partial_Q + 2iJ\partial_J\partial_L \\ &\quad + 2i\left(\frac{1}{L}J\partial_J + L\partial_L^2 + 4\partial_L\right) + \frac{40i}{91} \\ F &= J\partial_L + \frac{J}{L} - \frac{4i\sqrt{2}Q^3}{27L^2} \end{aligned} \quad (3.160)$$

It would be interesting to determine the lowest K -type in this polarization. This construction was generalized to all quaternionic-Kähler symmetric spaces in [41], where it was argued that the generalized topological string amplitude in this polarization computes the degeneracies of five-dimensional black holes with electric charge Q , angular momentum J and NUT charge L .

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